1. Question: Convergence of Sequences (*elementary*)

1.1. Show that $\lim_{n\to\infty} \dfrac{n^2}{n^2+n+1} = 1$ using proposition 6.1.

1.2. Prove the following statement: *If $\{x_n\}$ is convergent, then $\{x_n\}$ is bounded.*
Hint: Here, it might help you to set $\varepsilon = 1$ and separately consider the cases $n \leq M$ and $n > M$ for some $M \in \mathbb{N}_{>0}$.

1.3. In each of the following cases, decide whether the sequence is convergent or divergent. If convergent, find its limit. You may use the following fact:

$$\textit{If } c \in (0,1), \textit{ then } \underset{n\to\infty}{c^n} = 0. \textit{ If } c > 1, \textit{ then } \{c_n\} \textit{ is unbounded and diverges.}$$

(i) $a_n = 5 - 0.1^n$
(ii) $a_n = 1^n + (-1)^n$
(iii) $a_n = \dfrac{\sin n}{n}$
(iv) $a_n = \dfrac{2-n}{7+3n}$
(v) $a_n = \dfrac{3^{n+1}}{2^{2n+1}}$
(vi) $a_n = \dfrac{3^{n-1}}{2^{n+3}}$

Solution:

1.1. We have

$$\left|\frac{n^2}{n^2+n+1} - 1\right| = \left|\frac{-n-1}{n^2+n+1}\right| = \frac{n+1}{n^2+n+1} \leq \frac{n+1}{n^2+n} = \frac{1}{n}.$$

Thus,

$$0 \leq \left|\frac{n^2}{n^2+n+1} - 1\right| \leq \frac{1}{n} \to 0 \Longrightarrow \lim_{n\to\infty}\left|\frac{n^2}{n^2+n+1} - 1\right| = 0$$

1.2. Suppose that $\lim_{n\to\infty} x_n = x$. Thus, there exists an $M \in \mathbb{N}$ such that $|x_n - x| < 1$ for all $n \geq M$. Let

$$B = \max\{|x_1|, |x_2|, \ldots, |x_{M-1}|, |x| + 1\}$$

If $n < M$, then $|x_n| \leq B$ by construction. If $n \geq M$, then

$$|x_n| \leq |x_n - x| + |x| < 1 + |x| \leq B.$$

1.3. (i) We separate the two parts via the difference rule.

$$\lim_{n\to\infty} 5 - 0.1^n = \lim_{n\to\infty} 5 - \lim_{n\to\infty} 0.1^n$$

The limit of the constant sequence is that same value. Since $0.1 < 1, 0.1^n$ approaches zero as n gets large.

$$\lim_{n\to\infty} 5 - 0.1^n = \lim_{n\to\infty} 5 - \lim_{n\to\infty} 0.1^n = 5 - 0 = 5$$

(ii) The first few terms of the sequence are: $2, 0, 2, 0, 2, 0, 2, 0, 2, 0\ldots$ This sequence diverges.

(iii) This sequence converges to zero by the sandwich rule.

$$\lim_{n\to\infty}\left(-\frac{1}{n}\right) = 0 \quad \text{and} \quad \lim_{n\to\infty}\left(\frac{1}{n}\right) = 0$$

Since $-1 \leq \sin n \leq 1, \quad -\frac{1}{n} \leq \frac{\sin n}{n} \leq \frac{1}{n}$. Thus $\lim_{n\to\infty} \frac{\sin n}{n} = 0$.

(iv)

$$\lim_{n\to\infty}\frac{2-n}{7+3n}=\lim_{n\to\infty}\frac{\frac{2}{n}-1}{\frac{7}{n}+3}=\frac{\lim_{n\to\infty}\frac{2}{n}-\lim_{n\to\infty}1}{\lim_{n\to\infty}\frac{7}{n}+\lim_{n\to\infty}3}=\frac{0-1}{0+3}=-\frac{1}{3}$$

(v)

$$\lim_{n\to\infty}\frac{3^{n+1}}{2^{2n+1}}=\lim_{n\to\infty}\frac{3\cdot 3^n}{2\cdot 2^{2n}}=\frac{3}{2}\lim_{n\to\infty}\frac{3^n}{(2^2)^n}=\frac{3}{2}\lim_{n\to\infty}\frac{3^n}{4^n}=\frac{3}{2}\lim_{n\to\infty}\left(\frac{3}{4}\right)^n$$

Since $\frac{3}{4}<1$, $\left(\frac{3}{4}\right)^n$ approaches 1 as n gets large. Thus

$$\frac{3}{2}\lim_{n\to\infty}\left(\frac{3}{4}\right)^n=\frac{3}{2}\cdot 0=0$$

(vi)

$$\lim_{n\to\infty}\frac{3^{n-1}}{2^{n+3}}=\lim_{n\to\infty}\frac{\frac{1}{3}\cdot 3^n}{8\cdot 2^n}=\frac{1}{24}\lim_{n\to\infty}\frac{3^n}{2^n}=\frac{1}{24}\lim_{n\to\infty}\left(\frac{3}{2}\right)^n$$

Since $\frac{3}{2}>1$, $\left(\frac{3}{2}\right)^n$ diverges to infinity.

2. Question: Convergence of series *(elementary)*

2.1. Prove the following statement: *If $|r|\geq 1$, then $\sum_{n=0}^{\infty} r^n$ diverges.*

2.2. *Using the result from **2.1** and the identity*

$$\sum_{n=0}^{m} r^n=\frac{1-r^{m+1}}{1-r},$$

*prove that the so-called **geometric series** $\sum_{n=0}^{\infty}\alpha(r)^n$, with some scalar α, converges to $\frac{\alpha}{1-r}$ if and only if $|r|<1$.*

2.3. In each of the following cases, decide whether the series is convergent or divergent. If convergent, find its limit.

(i) $\sum_{n=0}^{\infty}\frac{2}{3^n}$

(ii) $\sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^{n+1}$

(iii) $\sum_{n=1}^{\infty}\frac{1}{n(n+1)}$

(iv) $\sum_{n=1}^{\infty}\frac{1}{n}$ (*Attention: The answer might not be what you think*)

Solution:

2.1. If $|r|\geq 1$, then $\lim_{m\to\infty} r^m\neq 0$. Therefore, $\sum_{n=0}^{\infty} r^n$ diverges, and if this wasn't the case then $\lim_{m\to\infty} r^m=0$ by proposition 6.2 which is a contradiction.

2.2. We start by writing the geometric series as a sequence $\{\sum_{n=0}^{m}\alpha(r)^n\}_{m=1}^{\infty}$. From the calculation rule for convergence of sequences, it follows that

$$\lim_{m\to\infty}\sum_{n=0}^{m}\alpha(r)^n=\alpha\lim_{m\to\infty}\sum_{n=0}^{m}(r)^n\,, \qquad (\star)$$

which clearly diverges if $|r|\geq 1$ from **2.1**.

Furthermore, we have

$$\lim_{m\to\infty}\sum_{n=0}^{m} r^n = \lim_{m\to\infty}\frac{1-r^{m+1}}{1-r} \overset{\text{by the hint from } \mathbf{1.3}}{=} \frac{1-0}{1-r} = \frac{1}{1-r}.$$

Finally, using the same reasoning as in ($\star$), we get

$$\lim_{m\to\infty}\sum_{n=0}^{m}\alpha(r)^n = \frac{\alpha}{1-r},$$

proving the statement.

2.3.

(i) This is a geometric series with $\alpha = 2$ and $r = \frac{1}{3}$. The sum of the series exists and can be computed as

$$\frac{\alpha}{1-r} = \frac{2}{1-\frac{1}{3}} = \frac{2}{\frac{2}{3}} = 3.$$

(ii) Again, this is a geometric sequence, this time with $\alpha = \frac{2}{3}$ and $r = \frac{2}{3}$. The sum of the series exists and can be computed as

$$\frac{\alpha}{1-r} = \frac{\frac{2}{3}}{1-\frac{2}{3}} = \frac{\frac{2}{3}}{\frac{1}{3}} = 2.$$

(iii) This series converges, which we can show directly by considering the sequence $\left\{\sum_{n=1}^{m}\frac{1}{n(n+1)}\right\}_{m=1}^{\infty}$:

$$\begin{aligned}\sum_{n=1}^{m}\frac{1}{n(n+1)} &= \sum_{n=1}^{m}\frac{1}{n} - \frac{1}{n+1}\\ &= \left(1+\frac{1}{2}+\dots+\frac{1}{m}\right) - \left(\frac{1}{2}+\dots\frac{1}{m}+\frac{1}{m+1}\right)\\ &= 1-\frac{1}{m+1} \quad \overset{m\to\infty}{\longrightarrow} \quad 1-0 = 1\,.\end{aligned}$$

Therefore, we have

$$\lim_{m\to\infty}\sum_{n=1}^{\infty}\frac{1}{n(n+1)} = 1\,.$$

(iv) $\sum_{n=1}^{\infty}\dfrac{1}{n}$ is called the *harmonic series*. It diverges, for which there are different proofs. The following by Leo Goldmakher is the shortest one I found:

Suppose the harmonic series converges to H, i.e.

$$H = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\cdots$$

Then

$$\begin{aligned}H &\geq 1+\frac{1}{2}+\underbrace{\frac{1}{4}+\frac{1}{4}}+\underbrace{\frac{1}{6}+\frac{1}{6}}+\underbrace{8}_{\frac{1}{8}+\frac{1}{8}}+\cdots\\ &= 1+\frac{1}{2}+\frac{1}{2}+\cdots\\ &= \frac{1}{2}+H.\end{aligned}$$

This contradiction concludes the proof.

3. Question: Limits inferior and superior (*elementary*)

3.1. Calculate the limit superior and limit inferior for the following sequences.

(a) $x_n = \dfrac{1}{n}$

(b) $x_n = (-1)^n$

3.2. Do the sequences (a) and (b) from above question converge? If so, name their limit.

3.3. The following statement is an important fact:

Let $\{x_n\}$ be a bounded sequence. Then, $\{x_n\}$ converges if and only if $\liminf x_n = \limsup x_n$.

Direction ($\Longrightarrow$) may be proven via subsequences (see the lecture named under *Helpful Additional Resources*). Prove the other direction, i.e. $\liminf x_n = \limsup x_n \Longrightarrow \{x_n\}$ converges.

Solution:

3.1.

(a) We may do this directly:

$$\sup\{1/k \mid k \geq n\} = \frac{1}{n} \to 0 \Longrightarrow \limsup_{n\to\infty} x_n = 0$$
$$\inf\{1/k \mid k \geq n\} = 0 \to 0 \Longrightarrow \liminf_{n\to\infty} x_n = 0$$

(b) Notice that $\{(-1)^k \mid l \geq n\} = \{-1, 1\}$. Thus, the supremum of these sets is always 1 and the infimum is always -1 . Therefore,

$$\limsup_{n\to\infty} x_n = 1 \text{ and } \liminf_{n\to\infty} x_n = -1$$

3.2. Clearly, the sequence $x_n = (-1)^n$ does not converge. Meanwhile, the sequence $x_n = \frac{1}{n}$ converges to 0 by the following:

Let $\epsilon > 0$. Choose $M \in \mathbb{N}$ such that $M^{-1} > \epsilon^{-1}$. Hence, for all $n \geq M$, $\left|\dfrac{1}{n} - 0\right| = \dfrac{1}{n} \leq \dfrac{1}{M} \leq \epsilon$.

Note that this perfectly aligned with the statement you are asked to partially prove next.

3.3. ($\Longleftarrow$) Suppose $\liminf x_n = \limsup x_n$. Then, $\forall n \in \mathbb{N}$,

$$\inf\{x_k \mid k \geq n\} \leq x_n \leq \sup\{x_k \mid k \geq n\}$$

By the Squeeze Theorem, since $\lim_{k\to\infty} \inf\{x_k \mid k \geq n\} = \lim_{k\to\infty} \sup\{x_k \mid k \geq n\}$ by assumption, we have

$$\lim_{n\to\infty} x_n = \liminf x_n = \limsup x_n.$$

Therefore, x_n converges.

4. Question: Differentiability of real-valued functions (*elementary*)

4.1. Prove that for the function $f(x) = ax + b$,

$$f'(c) = a \quad \forall c \in \mathbb{R}.$$

4.2. Is the function

$$f : \mathbb{R} \to \mathbb{R}, \quad x \mapsto \begin{cases} 1/x & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

differentiable at $x \neq 0$? What about $x = 0$?

4.3. Use the Mean Value Theorem to prove the following statement:
If $f : I \to \mathbb{R}$ is differentiable and $f'(x) = 0$ for all $x \in I$, then f is constant.

Solution:

4.1. This follows as

$$\lim_{x\to c} \frac{f(x) - f(c)}{x - c} = \lim_{x\to c} \frac{ax + b - (ac + b)}{x - c} = a \lim_{x\to c} \frac{x - c}{x - c} = \lim_{x\to c} a = a.$$

4.2. f is differentiable at $x \neq 0$ with derivative $f'(x) = -1/x^2$ since

$$\begin{aligned} \lim_{h\to 0} \left[\frac{f(c+h) - f(c)}{h}\right] &= \lim_{h\to 0} \left[\frac{1/(c+h) - 1/c}{h}\right] \\ &= \lim_{h\to 0} \left[\frac{c - (c+h)}{hc(c+h)}\right] \\ &= -\lim_{h\to 0} \frac{1}{c(c+h)} \\ &= -\frac{1}{c^2}. \end{aligned}$$

However, f is not differentiable at 0 since the limit

$$\lim_{h\to 0} \left[\frac{f(h) - f(0)}{h}\right] = \lim_{h\to 0} \left[\frac{1/h - 0}{h}\right] = \lim_{h\to 0} \frac{1}{h^2}$$

does not exist.

4.3. Let $a, b \in I$ with $a < b$. Then, f is continuous on $[a, b]$ and differentiable on (a, b). Therefore, $\exists c \in (a, b)$ such that $f(b) - f(a) = (b - a)f'(c) = 0$. Hence, $f(b) = f(a)$ for all $a, b \in I$ such that $a < b$.

5. Question: Differentiability and Continuity (*slightly advanced*)

Recall that we already defined the concept of continuous functions in the first Tutorial session using the **epsilon-delta criterion**. For a function $f : S \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ we can re-write it as follows:

The function f is continuous on S if

$$\forall c \in S \text{ and } \forall \epsilon > 0, \exists \delta = \delta(\epsilon, c) > 0 \text{ such that } \forall x \in S, |x - c| < \delta \Longrightarrow |f(x) - f(c)| < \epsilon.$$

Here, $\delta(\epsilon, c)$ denotes the fact that δ can depend on ϵ and c.

5.1. Does a function $f : S \subseteq \mathbb{R} \to \mathbb{R}$ being continuous imply that f is differentiable? Prove your answer using the function $f(x) := |x|$.
Hint: You might need the reverse triangle inequality: $||x| - |x_0|| \leq |x - x_0|$.

5.2. Show that the converse it true, i.e. if $f : S \subseteq \mathbb{R} \to \mathbb{R}$ is differentiable at $c \in S$, then f is continuous at c.

Solution:

5.1. No. We can prove this by giving a counter example, namely $f(x) := |x|$ (also shown in an Illustration from the Booklet).

Step 1: Prove continuity.
Let $x_0 \in \mathbb{R}$ be arbitrary. Let $\epsilon > 0$. Let $\delta = \epsilon > 0$. Then for any $x \in \mathbb{R}$ with $0 < |x - x_0| < \delta$, we trivially have $||x| - |x_0|| \leq |x - x_0| < \epsilon$ by our choice of $\delta = \epsilon$.

Step 2: Prove non-differentiability.
We find a sequence $x_n \to 0$ such that $\lim_{n\to\infty} \frac{f(x_n)-f(0)}{x_n-0}$ does not exist. Let $x_n = \frac{(-1)^n}{n}$. Then, $\lim_{n\to\infty} x_n = 0$. However,

$$\frac{f\left(x_n\right)-f(0)}{x_n-0}=\frac{\left|(-1)^n / n\right|}{(-1)^n / n}=(-1)^n$$

and $\lim_{n\to\infty}(-1)^n$ does not exist.

5.2. f is continuous at $c \in S \iff \lim_{x\to c} f(x) = f(c)$. Now,

$$\begin{aligned}
\lim_{x\to c} f(x) &= \lim_{x\to c}(f(x) - f(c) + f(c)) \\
&= \lim_{x\to c}\left((x-c)\frac{f(x)-f(c)}{x-c} + f(c)\right) \\
&= 0 \cdot f'(c) + f(c) = f(c)\,.
\end{aligned}$$

If you have any questions or feedback, please feel free to contact me via E-mail at hannah.kuempel@stat.uni-muenchen.de!!

Also, thank you Dr. Casey Rodriguez and Marta Hidegkuti whose exercises this sheet was inspired by.