1. Question: Determinants (*elementary*)

1.1. What are the determinants of the following matrices

$$A = \begin{bmatrix} 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}?$$

1.2. Compute the determinants of

$$A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 3 & 1 & 4 & 5 & 6 \\ 0 & 3 & 2 & 7 & 1 & 8 \\ 0 & 0 & 1 & 2 & 4 & 3 \\ 0 & 0 & 0 & 1 & 5 & 2 \\ 0 & 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix} .$$

1.3. Use a cofactor expansion across the third row to compute $\det(A)$, where

$$A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$$

1.4. Compute $\det(A)$, where

$$A = \begin{bmatrix} 3 & -7 & 8 & 9 & -6 \\ 0 & 2 & -5 & 7 & 3 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 2 & 4 & -1 \\ 0 & 0 & 0 & -2 & 0 \end{bmatrix}$$

Solution:

1.1. Per definition, $|A| = 4$. Furthermore,

$$|B| = (3 \cdot 2) - (1 \cdot 5) = 1 .$$

1.2. $\det A = a_{11} \det A_{11} - a_{12} \det A_{12} + a_{13} \det A_{13}$:

$$\begin{aligned} \det A &= 1 \cdot \det \begin{bmatrix} 4 & -1 \\ -2 & 0 \end{bmatrix} - 5 \cdot \det \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix} + 0 \cdot \det \begin{bmatrix} 2 & 4 \\ 0 & -2 \end{bmatrix} \\ &= 1(0-2) - 5(0-0) + 0(-4-0) = -2 . \end{aligned}$$

Furthermore, since B is an upper triangular matrix,

$$\det(B) = 2 \cdot 3 \cdot 1 \cdot 1 \cdot 1 \cdot 3 = 18 .$$

1.3. Compute

$$\begin{aligned} \det A &= a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33} \\ &= (-1)^{3+1} a_{31} \det A_{31} + (-1)^{3+2} a_{32} \det A_{32} + (-1)^{3+3} a_{33} \det A_{33} \\ &= 0 \begin{vmatrix} 5 & 0 \\ 4 & -1 \end{vmatrix} - (-2) \begin{vmatrix} 1 & 0 \\ 2 & -1 \end{vmatrix} + 0 \begin{vmatrix} 1 & 5 \\ 2 & 4 \end{vmatrix} \\ &= 0 + 2(-1) + 0 = -2 \end{aligned}$$

1.4. The cofactor expansion down the first column of A has all terms equal to zero except the first. Thus

$$\det A = 3 \cdot \begin{vmatrix} 2 & -5 & 7 & 3 \\ 0 & 1 & 5 & 0 \\ 0 & 2 & 4 & -1 \\ 0 & 0 & -2 & 0 \end{vmatrix} + 0 \cdot C_{21} + 0 \cdot C_{31} + 0 \cdot C_{41} + 0 \cdot C_{51}$$

Henceforth we will omit the zero terms in the cofactor expansion. Next, expand this 4×4 determinant down the first column, in order to take advantage of the zeros there. We have

$$\det A = 3\cdot 2\cdot \begin{vmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{vmatrix}$$

This 3×3 determinant was computed in **1.2/1.3** and found to equal -2.
Hence $\det A = 3\cdot 2\cdot(-2) = -12$.

2. Question: Eigenspaces

2.1. Name all eigenvalues and corresponding eigenspaces of the identity matrix $\mathbf{I}_n$ in $\mathbb{R}^{n\times n}$, $n\in\mathbb{N}_{>0}$. (***elementary***)

2.2. Compute all eigenvalues and corresponding eigenspaces of the following matrices: (***basic***)

$$\boldsymbol{A} = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 4 & 0 & 1 \\ -1 & -6 & -2 \\ 5 & 0 & 0 \end{bmatrix}$$

2.3. Consider a vector $v\in\mathbb{R}^n$, $n\in\mathbb{N}_{>0}$. Name all eigenvectors and corresponding eigenspaces of $vv^\top$. (***basic***)

Solution:

2.1. Of course, for any vector $v\in\mathbb{R}^n$, we have that $\mathbf{I}_n v = v$. Since this immediately implies that $\mathbf{I}_n v = 1v$, so the only eigenvalue is $\lambda = 1$, with all vectors in $\mathbb{R}^n$ corresponding to it. Therefore, the eigenspace is $E_1 = \mathbb{R}^n$.

2.2. <u>For matrix A</u>:

Step 0: Characteristic Polynomial. From our definition of the eigenvector $\boldsymbol{x}\neq\mathbf{0}$ and eigenvalue λ of $\boldsymbol{A}$, there will be a vector such that $\boldsymbol{A}\boldsymbol{x} = \lambda\boldsymbol{x}$, i.e., $(\boldsymbol{A}-\lambda\boldsymbol{I})\boldsymbol{x} = \mathbf{0}$. Since $\boldsymbol{x}\neq\mathbf{0}$, this requires that the kernel (null space) of $\boldsymbol{A}-\lambda\boldsymbol{I}$ contains more elements than just $\mathbf{0}$. This means that $\boldsymbol{A}-\lambda\boldsymbol{I}$ is not invertible and therefore $\det(\boldsymbol{A}-\lambda\boldsymbol{I}) = 0$. Hence, we need to compute the roots of the characteristic polynomial to find the eigenvalues.

Step 1: Eigenvalues. The characteristic polynomial is

$$\begin{aligned} p_{\boldsymbol{A}}(\lambda) &= \det(\boldsymbol{A}-\lambda\boldsymbol{I}) \\ &= \det\left(\begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = \begin{vmatrix} 4-\lambda & 2 \\ 1 & 3-\lambda \end{vmatrix} \\ &= (4-\lambda)(3-\lambda) - 2\cdot 1 \end{aligned}$$

We factorize the characteristic polynomial and obtain

$$p(\lambda) = (4-\lambda)(3-\lambda) - 2\cdot 1 = 10 - 7\lambda + \lambda^2 = (2-\lambda)(5-\lambda)$$

giving the roots $\lambda_1 = 2$ and $\lambda_2 = 5$.

Steps 2,3: Eigenvectors and Eigenspaces. We find the eigenvectors that correspond to these eigenvalues by looking at vectors $\boldsymbol{x}$ such that

$$\begin{bmatrix} 4-\lambda & 2 \\ 1 & 3-\lambda \end{bmatrix}\boldsymbol{x} = \mathbf{0}$$

For $\lambda = 5$ we obtain

$$\begin{bmatrix} 4-5 & 2 \\ 1 & 3-5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{0}.$$

We solve this homogeneous system and obtain a solution space

$$E_5 = \text{span}\left[\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right]$$

This eigenspace is one-dimensional as it possesses a single basis vector. Analogously, we find the eigenvector for $\lambda = 2$ by solving the homogeneous system of equations

$$\begin{bmatrix} 4-2 & 2 \\ 1 & 3-2 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} \boldsymbol{x} = \mathbf{0}.$$

This means any vector $\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, where $x_2 = -x_1$, such as $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, is an eigenvector with eigenvalue 2 . The corresponding eigenspace is given as

$$E_2 = \text{span}\left[\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right]$$

For matrix B:

Step 1: We can rewrite the characteristic polynomial as follows

$$\begin{aligned}
& p_B(\lambda) = \det(B - \lambda \boldsymbol{I}) && \overset{!}{=} 0 \\
\Longrightarrow\ & \begin{vmatrix} \lambda-4 & 0 & 0 \\ 1 & \lambda+6 & 2 \\ -5 & 0 & \lambda \end{vmatrix} && \overset{!}{=} 0 \\
\Longrightarrow\ & (\lambda-4)((\lambda+6)(\lambda)-0) - 1(0-(-5)(\lambda+6)) && \overset{!}{=} 0 \\
\Longrightarrow\ & (\lambda-4)((\lambda+6)(\lambda)) - 5(\lambda+6) && \overset{!}{=} 0 \\
\Longrightarrow\ & (\lambda+6)(\lambda(\lambda-4)-5) = (\lambda+6)\left(\lambda^2 - 4\lambda - 5\right) && \overset{!}{=} 0 \\
\Longrightarrow\ & (\lambda+6)(\lambda-5)(\lambda+1) && \overset{!}{=} 0
\end{aligned}$$

From which it immediately follows that the eigenvalues are

$$\lambda_1 = -6, \quad \lambda_2 = 5, \text{ and } \lambda_3 = -1\,.$$

Step 2: Eigenvectors.
For $\lambda_1 = -6$, we get

$$\left[\begin{array}{ccc|c} 4-(-6) & 0 & 1 & 0 \\ -1 & -6-(-6) & -2 & 0 \\ 5 & 0 & 0-(-6) & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Solving this gives us the corresponding eigenvector $v_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ (other values than 1 are of course correct, too!)

For $\lambda_2 = 5$, we get

$$\left[\begin{array}{ccc|c} 4-5 & 0 & 1 & 0 \\ -1 & -6-5 & -2 & 0 \\ 5 & 0 & 0-5 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & \frac{3}{11} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Solving this gives us the corresponding eigenvector $v_1 = \begin{pmatrix} 1 \\ -\frac{3}{11} \\ 1 \end{pmatrix}$.

For $\lambda_3 = -1$, we get

$$\left[\begin{array}{ccc|c} 4-(-1) & 0 & 1 & 0 \\ -1 & -6-(-1) & -2 & 0 \\ 5 & 0 & 0-(-1) & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{5} & 0 \\ 0 & 1 & \frac{9}{25} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Solving this gives us the corresponding eigenvector $v_1 = \begin{pmatrix} -\frac{1}{5} \\ -\frac{9}{25} \\ 1 \end{pmatrix}$.

Step 3: Lastly, we get the following eigenspaces from step 2:

$$E_{-6} = \operatorname{span}\left\{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right\}, \quad E_5 = \operatorname{span}\left\{\begin{pmatrix} 1 \\ -\frac{3}{11} \\ 1 \end{pmatrix}\right\}, \quad \text{and} \quad E_{-1} = \operatorname{span}\left\{\begin{pmatrix} -\frac{1}{5} \\ -\frac{9}{25} \\ 1 \end{pmatrix}\right\}.$$

2.3. Consider another vector $x \in \mathbb{R}^n$. First, note that both $(vv^\top)x$ and $v(v^\top x)$ are vectors in $\mathbb{R}^n$. More specifically, for any $i \in \{1, \dots, n\}$ we have

$$[(vv^\top)x]_i = \sum_{j=1}^{n}(v_i v_j)x_j = v_i \sum_{j=1}^{n}(v_j x_j) = [v(v^\top x)]_i$$

So it immediately follows that

$$(vv^\top)x = v(v^\top x) \overset{(v^\top x) \text{ is a scalar}}{=} (v^\top x)v\,.$$

Therefore, the matrix has the following two eigenvalues with corresponding eigenspaces:

- $v^\top v$ with $E_{v^\top v} = \operatorname{span}(v)$ (this follows immediately from setting $x = v$)
- 0 with $E_0 = \ker(v^\top)$, i.e. all vectors $p \in \mathbb{R}^n$ so that $v^\top p = 0$.

3. Question: Conceptual questions *(basic)*

For some questions in this exercise, you will need the following: *The determinant of a square matrix is equal to the product of its eigenvalues, while the trace is equal to the sum of its eigenvalues.* (For a proof, see page 3 of `https://www.adelaide.edu.au/mathslearning/ua/media/120/evalue-magic-tricks-handout.pdf`)

3.1. Prove that a square matrix A is invertible if and only if it has no eigenvalues of value 0.

3.2. If an invertible matrix A has eigenvalues $\lambda_1, \dots, \lambda_n$ then what are the eigenvalues of A^{-1}? What about $A + I$? What about $A^\top$? What about A^2? What about A^3? Answer the same question for eigenvectors.

3.3. Is it possible to determine the eigenvalues of the following matrices?

(i) A, a 2×2 matrix with $\det(A) = -2$ and $\operatorname{tr}(A) = 0$.

(ii) B, a 3×3 matrix with $\det(A) = -2$ and $\operatorname{tr}(A) = 0$.

3.4. Prove that if matrices A and B are similar then they have identical eigenvalues.

3.5. Prove the following statement: *Similar matrices are always equivalent. However, equivalent matrices are not necessarily similar.*

Solution:

3.1. If there is an eigenvalue of 0 then $0 = \det(A - \lambda I) = \det(A - 0) = \det(A)$ so A is not invertible. If A is not invertible read the equalities in the opposite order and A has an eigenvalue of value 0.

3.2. A^{-1} will have reciprocal eigenvalues $\frac{1}{\lambda_1}, \ldots, \frac{1}{\lambda_n}$ with the same eigenvectors as A. (note this is well defined since none of these are zero from the previous problem). To see this if λ is an eigenvalue of A with eigenvector $\vec{v}$ then

$$A\vec{v} = \lambda\vec{v} \Longrightarrow A^{-1}A\vec{v} = A^{-1}\lambda\vec{v} \Longrightarrow \vec{v} = \lambda A^{-1}\vec{v} \Longrightarrow \frac{1}{\lambda}\vec{v} = A^{-1}\vec{v}$$

which shows $\vec{v}$ is an eigenvalue of A^{-1} with eigenvalue $\frac{1}{\lambda}$. A^T has the same eigenvalues as A. Once again if λ is an eigenvalue of A then

$$\det(A - \lambda I) = 0 \Longrightarrow \det(A - \lambda I)^T = 0 \Longrightarrow \det\left(A^T - \lambda I^T\right) = 0 \Longrightarrow \det\left(A^T - \lambda I\right) = 0$$

which shows that λ is an eigenvalue of A^T. The eigenvector will be different though in general. $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ has eigvenvector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ while $A^T = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ has eigenvector $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. A^n for a whole number $n \geq 1$ has the same eigenvalues which are the n^{th} power of those of A and the same eigenvectors as A. Let $\vec{v}$ be an eigenvector of A with eigenvalue λ. This means that $A\vec{v} = \lambda\vec{v}$. As a result

$$A^n\vec{v} = A^{n-1}A\vec{v} = A^{n-1}\lambda\vec{v} = \lambda A^{n-2}A\vec{v} = \lambda A^{n-2}\lambda\vec{v} = \ldots = \lambda^n\vec{v}$$

$A + I$ has the same eigenvectors as A but with eigenvalues $\lambda + 1$. If $\vec{v}$ is an eigenvector of A then $A\vec{v} = \lambda\vec{v}$ so

$$(A + I)\vec{v} = A\vec{v} + I\vec{v} = \lambda\vec{v} + \vec{v} = (\lambda + 1)\vec{v}$$

so $\vec{v}$ is an eigenvector with eigenvalue $\lambda + 1$.

3.3. (i) We know that $\det(A) = \lambda_1\lambda_2 = -2$ and $\text{tr}(A) = \lambda_1 + \lambda_2 = 0$. From the second we have $\lambda_1 = -\lambda_2$ which in the first gives

$$-\lambda_2^2 = -2 \Longrightarrow \lambda_2 = \sqrt{2}$$

which then forces $\lambda_1 = -\sqrt{2}$.

(ii) We can set this up similar to the previous problem, but we would have an extra variable λ_3 that would be "free" so we are going to have infinitely many solutions. $\lambda_1 = -2, \lambda_2 = 1, \lambda_3 = 1$ is one and $\lambda_1 = 1, \lambda_2 = 1, \lambda_3 = -2$ is another.

3.4. Suppose that λ is an eigenvalue of A. Then it follows that (i) $\det(A - \lambda I) = 0$.

Because B is similar to A there is an invertible matrix V such that $V^{-1}AV = B$. Multiply (i) on the left by $\det\left(V^{-1}\right)$ and on the right by $\det(V)$.

$$\begin{aligned}
\det\left(V^{-1}\right)\det(A - \lambda I)\det(V) &= 0 \\
\det\left(V^{-1}AV - \lambda V^{-1}IV\right) &= 0 \\
\det(B - \lambda I) &= 0
\end{aligned}$$

so that λ is an eigenvalue of B. As a result we see that every eigenvalue of A is an eigenvalue of B . The reverse is also true because of the symmetry of the statement.

3.5. For matrices A and B to be equivalent we need to find invertible matrices P, Q so that $A = P^{-1}BQ$. But since A ad B are similar, there is an invertible matrix S such that A= $S^{-1}BS$. So we can just

take $P, Q = S$.
Hoever, equivalent matrices are not necessarily similar, which we can prove by counterexample:

$$\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

are equivalent since they are achievable from each other by row operations. However, they are not similar since they have different determinants and thus different eigenvalues.

4. Question: Matrix decomposition (*basic*)

In this question, you may use the following: *A square matrix is called* ***orthogonal****, if its columns are*

- *orthonormal to each other (i.e. the product of any two columns is* 0*)*
- *of unit length (have length* 1*).*

For an orthogonal matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$*, it holds that* $\mathbf{A}^{-1} = \mathbf{A}^\top$.

4.1. Compute the eigendecomposition of $\boldsymbol{A} = \frac{1}{2}\begin{bmatrix} 5 & -2 \\ -2 & 5 \end{bmatrix}$.

4.2. Another popular matric decomposition is the Cholesky decomposition. It says that any symmetric matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$ with only strictly positive eigenvalues may be decomposed as follows:

$$\boldsymbol{A} = \boldsymbol{L}\boldsymbol{L}^\top = \begin{pmatrix} l_{11} & \dots & 0 \\ \vdots & \ddots & \vdots \\ l_{n1} & \dots & l_{nn} \end{pmatrix} \begin{pmatrix} l_{11} & \dots & l_{1n} \\ \vdots & \ddots & \vdots \\ 0 & \dots & l_{nn} \end{pmatrix}.$$

For the following decomposition of a 3×3 matrix, write all l_{ij}s, in terms of the a_{ij}s, $i, j = 1, 2, 3$.

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{21} & a_{22} & a_{32} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \boldsymbol{L}\boldsymbol{L}^\top = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{bmatrix}.$$

4.3. Write out the matrix decompositions for matrices A and B from **2.2** (and check whether the decomposition truly results in A and B if you want to practice matrix multiplication).

Solution:

4.1.

Step 1: Find the eigenvalues of $\mathbf{A}$:
The characteristic polynomial of $\boldsymbol{A}$ is

$$\det(\boldsymbol{A} - \lambda\boldsymbol{I}) = \det\left(\begin{bmatrix} \frac{5}{2} - \lambda & -1 \\ -1 & \frac{5}{2} - \lambda \end{bmatrix}\right)$$
$$= \left(\frac{5}{2} - \lambda\right)^2 - 1 = \lambda^2 - 5\lambda + \frac{21}{4} = \left(\lambda - \frac{7}{2}\right)\left(\lambda - \frac{3}{2}\right)$$

Therefore, the eigenvalues of $\boldsymbol{A}$ are $\lambda_1 = \frac{7}{2}$ and $\lambda_2 = \frac{3}{2}$ (the roots of the characteristic polynomial)

Step 2: Find n linearly independent eigenvectors of A.
the associated (normalized) eigenvectors are obtained via

$$\boldsymbol{A}\boldsymbol{p}_1 = \frac{7}{2}\boldsymbol{p}_1, \quad \boldsymbol{A}\boldsymbol{p}_2 = \frac{3}{2}\boldsymbol{p}_2.$$

This yields

$$\boldsymbol{p}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad \boldsymbol{p}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Clearly, the eigenvectors $\boldsymbol{p}_1, \boldsymbol{p}_2$ are linearly independent (and, thereby, form a basis of $\mathbb{R}^2$). ✓

Steps 3,4: Construct the matrix $\boldsymbol{P}$ to diagonalize $\boldsymbol{A}$. We collect the eigenvectors of $\boldsymbol{A}$ in $\boldsymbol{P}$ so that

$$\boldsymbol{P} = [\boldsymbol{p}_1, \boldsymbol{p}_2] = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}.$$

We then obtain

$$\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \begin{bmatrix} \frac{7}{2} & 0 \\ 0 & \frac{3}{2} \end{bmatrix} = \boldsymbol{D}.$$

Equivalently, we get (exploiting that $\boldsymbol{P}^{-1} = \boldsymbol{P}^\top$ since the eigenvectors $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ in this example form an orthonogal basis)

$$\underbrace{\frac{1}{2}\begin{bmatrix} 5 & -2 \\ -2 & 5 \end{bmatrix}}_{\boldsymbol{A}} = \underbrace{\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}}_{\boldsymbol{P}} \underbrace{\begin{bmatrix} \frac{7}{2} & 0 \\ 0 & \frac{3}{2} \end{bmatrix}}_{\boldsymbol{D}} \underbrace{\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}}_{\boldsymbol{P}^{-1}}.$$

4.2. Multiplying out the right-hand side yields

$$\boldsymbol{A} = \begin{bmatrix} l_{11}^2 & l_{21}l_{11} & l_{31}l_{11} \\ l_{21}l_{11} & l_{21}^2 + l_{22}^2 & l_{31}l_{21} + l_{32}l_{22} \\ l_{31}l_{11} & l_{31}l_{21} + l_{32}l_{22} & l_{31}^2 + l_{32}^2 + l_{33}^2 \end{bmatrix}.$$

Clearly, there is a simple pattern in the diagonal elements l_{ii}:

$$l_{11} = \sqrt{a_{11}}, \quad l_{22} = \sqrt{a_{22} - l_{21}^2}, \quad l_{33} = \sqrt{a_{33} - (l_{31}^2 + l_{32}^2)}.$$

Similarly for the elements below the diagonal (l_{ij}, where $i > j$), there is also a repeating pattern:

$$l_{21} = \frac{1}{l_{11}}a_{21}, \quad l_{31} = \frac{1}{l_{11}}a_{31}, \quad l_{32} = \frac{1}{l_{22}}\left(a_{32} - l_{31}l_{21}\right).$$

Thus, we constructed the Cholesky decomposition for any symmetric, positive definite 3×3 matrix. The key realization is that we can backward calculate what the components l_{ij} for the L should be, given the values a_{ij} for $\boldsymbol{A}$ and previously computed values of l_{ij}.

If you have any questions or feedback, please feel free to contact me via E-mail at hannah.kuempel@stat.uni-muenchen.de!!

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