1. Question: Basic questions (*elementary*)

1.1. Think of **two different bases** of $\mathbb{R}^3$ that are not of the form $\left\{ \begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ b \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ c \end{pmatrix} \right\}$ for any $a, b, c \in \mathbb{R}$.

1.2. Determine the rank of the following 3 matrices:

$$A = \begin{pmatrix} 1 & 2 & 1 \\ -2 & -3 & 1 \\ 3 & 5 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \text{ and } \quad C = \begin{pmatrix} 1 & 2 & 5 \\ 0 & 4 & 8 \\ 2 & 3 & 8 \\ 5 & 2 & 9 \end{pmatrix}.$$

1.3. Find the basis of the vector subspace $U \subseteq \mathbb{R}^5$, spanned by the vectors

$$\boldsymbol{x}_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \\ -1 \\ -1 \end{bmatrix}, \quad \boldsymbol{x}_2 = \begin{bmatrix} 2 \\ -1 \\ 1 \\ 2 \\ -2 \end{bmatrix}, \quad \boldsymbol{x}_3 = \begin{bmatrix} 3 \\ -4 \\ 3 \\ 5 \\ -3 \end{bmatrix}, \quad \boldsymbol{x}_4 = \begin{bmatrix} -1 \\ 8 \\ -5 \\ -6 \\ 1 \end{bmatrix} \in \mathbb{R}^5.$$

1.4. Consider $V = \{(a_1, a_2) : a_1, a_2 \in \mathbb{R}\}$. Is V a vector space for the following definitions, where $a_1, a_2, b_1, b_2, c \in \mathbb{R}$, of addition and scalar multiplication? Justify your answer.

(i) $(a_1, a_2) + (b_1, b_2) := (a_1 + 2b_1, a_2 + 3b_2)$ and $c\,(a_1, a_2) := (ca_1, ca_2)$.

(ii) $(a_1, a_2) + (b_1, b_2) := (a_1 + b_1, a_2 + b_2)$ and $c\,(a_1, a_2) = \begin{cases} (0, 0) & \text{if } c = 0 \\ (ca_1, \frac{a_2}{c}) & \text{if } c \neq 0 \end{cases}$.

Solution:

1.1. We just need any 3 linearly independent vectors in $\mathbb{R}^3$.
Two possibilities are:

$$\mathcal{B}_1 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}, \quad \text{and} \quad \mathcal{B}_2 = \left\{ \begin{bmatrix} 0.5 \\ 0.8 \\ 0.4 \end{bmatrix}, \begin{bmatrix} 1.8 \\ 0.3 \\ 0.3 \end{bmatrix}, \begin{bmatrix} -2.2 \\ -1.3 \\ 3.5 \end{bmatrix} \right\}.$$

1.2.

- $\text{rank}(A) = 2$, because

$$\begin{bmatrix} 1 & 2 & 1 \\ -2 & -3 & 1 \\ 3 & 5 & 0 \end{bmatrix} \xrightarrow{2R_1 + R_2 \to R_2} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 3 & 5 & 0 \end{bmatrix} \xrightarrow{-3R_1 + R_3 \to R_3} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & -1 & -3 \end{bmatrix}$$
$$\xrightarrow{R_2 + R_3 \to R_3} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{-2R_2 + R_1 \to R_1} \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}.$$

- $\text{rank}(B) = 3$, because the first three rows are linearly independent, but the fourth row only has zero entries.
- $\text{rank}(C) = 2$, because the third column is a linear combination of the first two, specifically by writing

$$1 \cdot \begin{pmatrix} 1 \\ 0 \\ 2 \\ 5 \end{pmatrix} + 2 \cdot \begin{pmatrix} 2 \\ 4 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ 8 \\ 8 \\ 9 \end{pmatrix}.$$

1.3. We are interested in finding out which vectors $\boldsymbol{x}_1, \dots, \boldsymbol{x}_4$ are a basis for U. For this, we need to check whether $\boldsymbol{x}_1, \dots, \boldsymbol{x}_4$ are linearly independent. Therefore, we need to solve

$$\sum_{i=1}^{4} \lambda_i \boldsymbol{x}_i = \mathbf{0}$$

which leads to a homogeneous system of equations with matrix

$$[\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3, \boldsymbol{x}_4] = \begin{bmatrix} 1 & 2 & 3 & -1 \\ 2 & -1 & -4 & 8 \\ -1 & 1 & 3 & -5 \\ -1 & 2 & 5 & -6 \\ -1 & -2 & -3 & 1 \end{bmatrix}.$$

With the basic transformation rules for systems of linear equations, we obtain the row-echelon form

$$\begin{bmatrix} 1 & 2 & 3 & -1 \\ 2 & -1 & -4 & 8 \\ -1 & 1 & 3 & -5 \\ -1 & 2 & 5 & -6 \\ -1 & -2 & -3 & 1 \end{bmatrix} \rightsquigarrow \cdots \rightsquigarrow \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since the pivot columns indicate which set of vectors is linearly independent, we see from the row-echelon form that $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_4$ are linearly independent (because the system of linear equations $\lambda_1 \boldsymbol{x}_1 + \lambda_2 \boldsymbol{x}_2 + \lambda_4 \boldsymbol{x}_4 = \mathbf{0}$ can only be solved with $\lambda_1 = \lambda_2 = \lambda_4 = 0$. Meanwhile, column 3 of the row Echelon form equals 1.5 times the second column).
Therefore, $\{\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_4\}$ is a basis of U.

1.4. *(i)* No, because axiom (ii) fails: Let $v = (1, 0), w = (0, 1)$. Then

$$v + w = (1, 0) + (0, 1) = (1, 3) \neq (2, 1) = (0, 1) + (1, 0) = w + v$$

(ii) No, because axiom (viii) fails: Let $c, d \in \mathbb{R}$ and $(a_1, a_2) \in V$. Then

$$\begin{aligned}(c + d)\,(a_1, a_2) &= \left((c + d)a_1, \frac{a_2}{c + d}\right) \\ &\neq \left((c + d)a_1, \frac{a_2}{c} + \frac{a_2}{d}\right) \\ &= \left(ca_1 + da_1, \frac{a_2}{c} + \frac{a_2}{d}\right) \\ &= c\,(a_1, a_2) + d\,(a_1, a_2)\,.\end{aligned}$$

2. Question: Linear system of equations *(basic)*

Let A be the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 2 \\ 2 & 2 & 3 & 3 \end{bmatrix}$$

2.1. Determine if the system $A\mathbf{x} = \mathbf{0}$ has zero, one or infinitely many solutions, and compute a basis of the space of solutions.

2.2. Is it true that the system $A\mathbf{x} = \mathbf{b}$ has a solution for any $b \in \mathbb{R}^3$? If so, prove the statement, otherwise find a counterexample.

Hint: For $A\mathbf{x} = \mathbf{b}$ to have a solution, both A and the augmented matrix $[A|\mathbf{b}]$ need to be of the same rank.

2.3. Prove the following statement: *The null space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^n$. Equivalently, the set of all solutions to a system $A\mathbf{x} = \mathbf{0}$ of m homogeneous linear equations in n unknowns is a subspace of $\mathbb{R}^n$.* (***less basic***)

Solution:

2.1. We have to find $\ker(A)$. At first we row-reduce A:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 2 \\ 2 & 2 & 3 & 3 \end{bmatrix} \overset{R2-R1 \,\&\, R3-2R1}{\longrightarrow} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \overset{R3-R2}{\longrightarrow} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

There are two pivots and two free variables, therefore the system has infinitely many solutions. We choose x_2 and x_4 as free variables, and

- from $x_3 + x_4 = 0$ we get $x_3 = -x_4$;
- from $x_1 + x_2 + x_3 + x_4 = 0$ we get $x_1 = -x_2$.

Thus

$$\ker A = \left\{ \begin{bmatrix} -x_2 \\ x_2 \\ -x_4 \\ x_4 \end{bmatrix} \middle| \, x_2, x_4 \in \mathbb{R} \right\}$$

and a basis is

$$\left(\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right).$$

2.2. Let $b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ be a generic vector of $\mathbb{R}^3$. The system $Ax = b$ has a solution if and only if the matrix A and the complete matrix

$$\bar{A} = \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & b_1 \\ 1 & 1 & 2 & 2 & b_2 \\ 2 & 2 & 3 & 3 & b_3 \end{array}\right]$$

have the same rank. That happens if and only if the row reduced form of $\bar{A}$, which is

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & b_1 \\ 0 & 0 & 1 & 1 & b_2 - b_1 \\ 0 & 0 & 0 & 0 & b_3 - b_2 - b_1 \end{array}\right],$$

has not a pivot in the third column, i.e. $b_3 - b_2 - b_1 = 0$. Any vector b not satisfying this condition, for example

$$b = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

is a counterexample for the statement.

2.3. Certainly $\text{Ker}(A)$ is a subset of $\mathbb{R}^n$ because A has n columns. We must show that $\text{Ker}(A)$ satisfies the three properties of a subspace. Of course, $\mathbf{0}$ is in $\text{Ker}(A)$. Next, let $\mathbf{u}$ and $\mathbf{v}$ represent any two vectors in $\text{Ker}(A)$. Then

$$A\mathbf{u} = \mathbf{0} \quad \text{and} \quad A\mathbf{v} = \mathbf{0}$$

To show that $\mathbf{u}+\mathbf{v}$ is in $\text{Ker}(A)$, we must show that $A(\mathbf{u}+\mathbf{v}) = \mathbf{0}$. Using a property of matrix multiplication, compute

$$A(\mathbf{u}+\mathbf{v}) = A\mathbf{u} + A\mathbf{v} = \mathbf{0}+\mathbf{0} = \mathbf{0}$$

Thus $\mathbf{u}+\mathbf{v}$ is in $\text{Ker}(A)$, and $\text{Ker}(A)$ is closed under vector addition. Finally, if c is any scalar, then

$$A(c\mathbf{u}) = c(A\mathbf{u}) = c(\mathbf{0}) = \mathbf{0}$$

which shows that $c\mathbf{u}$ is in $\text{Ker}(A)$. Thus $\text{Ker}(A)$ is a subspace of $\mathbb{R}^n$.

3. Question: Intersection and Addition of Subspaces (*slightly more advanced*)

Let $V \subseteq \mathbb{R}^4$ be the subspace $V = \text{span}\,(v_1, v_2)$, where

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad v_2 = \begin{bmatrix} 1 \\ 3 \\ 1 \\ 0 \end{bmatrix}$$

and let $W \subseteq \mathbb{R}^4$ be the subspace given by the solutions of the system

$$\begin{cases} x_1 + x_2 + 2x_4 = 0 \\ 2x_1 + x_2 - x_3 = 0. \end{cases}$$

Find a basis of $V \cap W := \{u : u \in V \text{ AND } u \in W\}$ and a basis of $V + W := \{u = v + w : \; v \in V, w \in W\}$.
Hint: A vector $x \in \mathbb{R}^4$ belongs to V if and only if the two matrices $[v_1\ v_2]$ and $\big[[v_1\ v_2]\,\big|\,x\big]$ have the same rank.

Solution:
Given the hint, we know that a vector $(x_1, x_2, x_3, x_4)^\top$ belongs to V if and only if

$$\begin{bmatrix} 1 & 1 \\ 1 & 3 \\ 1 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \left[\begin{array}{cc|c} 1 & 1 & x_1 \\ 1 & 3 & x_2 \\ 1 & 1 & x_3 \\ 0 & 0 & x_4 \end{array}\right]$$

have the same rank. We row-reduce the second one, obtaining

$$\left[\begin{array}{cc|c} 1 & 1 & x_1 \\ 0 & 2 & x_2 - x_1 \\ 0 & 0 & x_3 - x_1 \\ 0 & 0 & x_4 \end{array}\right]$$

and this matrix must not have a pivot in the third column; therefore a set of equations for V is

$$\begin{cases} x_3 - x_1 = 0 \\ x_4 = 0. \end{cases} \tag{$\dagger$}$$

We recall that W was defined via

$$\begin{cases} x_1 + x_2 + 2x_4 = 0 \\ 2x_1 + x_2 - x_3 = 0. \end{cases} \tag{$*$}$$

Putting together the systems (†) and (∗) gives a system of equations for $V \cap W$:

$$\begin{cases} x_3 - x_1 = 0 \\ x_4 = 0 \\ x_1 + x_2 + 2x_4 = 0 \\ 2x_1 + x_2 - x_3 = 0 \end{cases}$$

with associated matrix

$$A = \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 2 \\ 2 & 1 & -1 & 0 \end{bmatrix}$$

The row reduced form of A is

$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{cases} x_1 - x_3 = 0 \\ x_2 + x_3 = 0 \\ x_4 = 0. \end{cases}$$

We keep x_3 as a free variable, thus we have $x_1 = x_3$ and $x_2 = -x_3$, i.e.

$$V \cap W = \left\{ \begin{bmatrix} x_3 \\ -x_3 \\ x_3 \\ 0 \end{bmatrix} \middle| \, x_3 \in \mathbb{R} \right\} = \text{span}\left(\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} \right)$$

and that is a basis of $V \cap W$. Now we turn to $V + W$. First of all, we compute a set of generators for W solving the system (∗) : if we proceed as above, we get

$$W = \left\{ \begin{bmatrix} x_3 + 2x_4 \\ -x_3 - 4x_4 \\ x_3 \\ x_4 \end{bmatrix} \middle| \, x_3, x_4 \in \mathbb{R} \right\} = \text{span}\left(\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -4 \\ 0 \\ 1 \end{bmatrix} \right).$$

Therefore

$$V + W = \text{span}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -4 \\ 0 \\ 1 \end{bmatrix} \right)$$

and we have to extract a set of linearly independent vectors. To do so, we put the generators in a matrix and we row-reduce it:

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 1 & 3 & -1 & -4 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 2 & -2 & -6 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The pivots are in the first, second and fourth columns, thus a basis of $V + W$ is

$$\left(\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -4 \\ 0 \\ 1 \end{bmatrix} \right).$$

4. Question: Linear Maps (*slightly more advanced*)

4.1. Let $f : \mathbb{R}^2 \to \mathbb{R}^3$ be the linear map defined as

$$f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x + 2y \\ 2x + 4y \\ x + ay \end{bmatrix}$$

where $a \in \mathbb{R}$ is a parameter. Find the matrix $[f]$ associated to f with respect to the standard bases of $\mathbb{R}^2$ and $\mathbb{R}^3$.

4.2. Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear map such that

$$f\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}, \quad f\left(\begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 6 \\ 8 \\ 10 \end{bmatrix} \quad \text{and} \quad f\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 10 \\ 14 \\ 18 \end{bmatrix}.$$

(i) Compute the dimensions of $\ker f$ and $\operatorname{Im} f$.

(ii) For f from **4.2.**, compute $f\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right)$.

Solution:

4.1. If $\left\{e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}$ is the standard basis of $\mathbb{R}^2$, the i-th column of $[f]$ is the vector of the coordinates of $f(e_i)$ with respect to the standard basis of $\mathbb{R}^3$. Since

$$f(e_1) = f\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \quad \text{and} \quad f(e_2) = f\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 4 \\ a \end{bmatrix}$$

we have

$$[f] = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 1 & a \end{bmatrix}.$$

4.2. We begin from question (b). Since f is linear, if $f(v) = w$ then $f(v/2) = w/2$, so we can compute

$$f\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = f\left(\frac{1}{2}\begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}\right) = \frac{1}{2} f\left(\begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix} 6 \\ 8 \\ 10 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}.$$

This means that we know the images of all the vectors in the standard basis of $\mathbb{R}^3$; these images are the columns of the matrix associated to f with respect to the standard basis:

$$[f] = \begin{bmatrix} 10 & 3 & 2 \\ 14 & 4 & 3 \\ 18 & 5 & 4 \end{bmatrix}.$$

Then

(i) We notice that

$$\begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \\ 9 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 10 \\ 14 \\ 18 \end{bmatrix},$$

i.e. the three columns of $[f]$ are linearly dependent. Thus the rank of $[f]$ is at most two-in fact, it is exactly 2 because the second and third columns are linearly independent. So we can conclude that $\dim(\mathrm{Im}(f)) = 2$ and

$$\dim \mathrm{Ker}(f) = \dim(\mathbb{R}^3) - \dim(\mathrm{Im}(f)) = 3 - 2 = 1$$

(ii) Applying this matrix to the vector $(1, 1, 1)^\top$ we get

$$f\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 10 & 3 & 2 \\ 14 & 4 & 3 \\ 18 & 5 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 10+3+2 \\ 14+4+3 \\ 18+5+4 \end{bmatrix} = \begin{bmatrix} 15 \\ 21 \\ 27 \end{bmatrix}.$$

5. Question: Vector space of real-valued functions *(Tedious, but definitely a fact to remember even if you don't want to verify)*

Prove that the set of all real-valued functions $f : \mathbb{R} \to \mathbb{R}$ is a vector space, if addition and scalar multiplication is defined as follows, for $c \in \mathbb{R}$:

$$(f + g)(x) = f(x) + g(x), \quad (cf)(x) = cf(x)\,.$$

Solution:
Here, vector space axioms (i) and (vi) immediately follow from the given definitions of addition and scalar multiplication. The verification of the remaining axioms, along with very neat visualizations, may be found on the following website: `https://thenumb.at/Functions-are-Vectors/#proofs`.

If you have any questions or feedback, please feel free to contact me via E-mail at hannah.kuempel@stat.uni-muenchen.de!!

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