1. Question: Intuition for those unfamiliar (*very elementary*)

1.1. Write out the transpose of the following matrix:

$$\mathbf{A} = \begin{pmatrix} 1 & 3 & 5 & 7 & 1 \\ 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 10 \end{pmatrix}$$

Solution:

We simply write the rows of $\mathbf{A}$ as the columns of $\mathbf{A}^\top$:

$$\mathbf{A}^T = \begin{pmatrix} 1 & 2 & 1 & 6 \\ 3 & 3 & 2 & 7 \\ 5 & 4 & 3 & 8 \\ 7 & 5 & 4 & 9 \\ 1 & 6 & 5 & 10 \end{pmatrix}$$

Notice that when $\mathbf{A} \in \mathbb{R}^{4\times 5}$, $\mathbf{A}^\top \in \mathbb{R}^{5\times 4}$!

1.2. Write out two matrices that cannot be multiplied with each other.

Solution:

There are many possible answers here. We just need two matrices so that the number of columns of the first matrix is not equal to the number of rows in the second. One example is

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 1 & 3 \\ 1 & 0 & 4 \\ 0 & 1 & 5 \end{bmatrix}.$$

1.3. Write 2×2 matrices A and B such that $AB \neq BA$. Verify your solution by computing the products.

Solution:

Almost any two matrices you write down will work. For instance, if

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

then

$$AB = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$$

but

$$BA = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$$

1.4. Write 2×2 matrices A, B, C such that $AB = AC$ but $B \neq C$. Verify your solution by computing the products.

Solution:

First, a helpful observation. Suppose A is invertible and $AB = AC$. Then multiplying both sides by A^{-1} gives $A^{-1}(AB) = A^{-1}(AC)$. Using distributivity and the fact that $A^{-1}A = I$, this implies that $IB = IC$ and therefore $B = C$. So if we want to solve this problem, we need to pick some A that is not invertible. The simplest option is

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

In which case, we can choose B and C to be any two matrices. For instance

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

Here's another example.

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad C = \begin{bmatrix} 10 & 11 \\ 0 & 1 \end{bmatrix}$$

1.5. Find two matrices that are inverse to each other.

Solution:

The easiest answer would of course be, for any $a_1, \ldots, a_n \in \mathbb{R} \setminus \{0\}$, $n \in \mathbb{N}_{>0}$, the following two diagonal matrices

$$\mathbf{A} = \text{diag}(a_1, \ldots, a_n) \quad \& \quad \mathbf{B} = \text{diag}\left(\frac{1}{a_1}, \ldots, \frac{1}{a_n}\right)$$

Another option would be the following two:

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 & 1 \\ 4 & 4 & 5 \\ 6 & 7 & 7 \end{bmatrix} \quad \& \quad \boldsymbol{B} = \begin{bmatrix} -7 & -7 & 6 \\ 2 & 1 & -1 \\ 4 & 5 & -4 \end{bmatrix}.$$

1.6. Find (and sketch) a system of linear equations each, that

- has infinitely many solutions
- has exactly one solution
- has no solution.

(Of course, you shouldn't use the examples from the booklet.)

Solution:

- The following is a system of linear equations with infinitely many solutions:

$$\begin{aligned} x + y &= 1 \\ 2x + 2y &= 2 \end{aligned}$$

- The following is a system of linear equations with exactly one solution:

$$x = 1$$
$$y = 2$$

- The following is a system of linear equations with no solution:

$$\begin{aligned} x + y &= 1 \\ 2x + 2y &= 3 \end{aligned}$$

The sketches are left to you, but the visual idea is of course the same as in the booklet.

2. Question: Vector and Matrix Multiplication (*elementary, but good computational practice!*)

2.1. Given the vectors $\mathbf{a}$ and $\mathbf{b}$:

$$\mathbf{a} = \begin{pmatrix} 1 \\ 0 \\ 5 \\ 7 \end{pmatrix} \quad \& \quad \mathbf{b} = \begin{pmatrix} 2 \\ 4 \\ 0 \\ 8 \end{pmatrix}$$

Compute the following:

(i) the four products $\mathbf{a}^T\mathbf{b}$; $\mathbf{b}^T\mathbf{a}$; $\mathbf{a}^T\mathbf{b}\mathbf{a}$; $\mathbf{b}^T\mathbf{a}\mathbf{b}$
(*Hint: This should only be three different values*)

(ii) the two products $\mathbf{a}\mathbf{b}^T$ and $\mathbf{b}\mathbf{a}^T$. Do you notice anything here?

Solution:

(i) First, we get

$$\mathbf{a}^T\mathbf{b} = \begin{pmatrix} 1 & 0 & 5 & 7 \end{pmatrix} \begin{pmatrix} 2 \\ 4 \\ 0 \\ 8 \end{pmatrix} = 1 \cdot 2 + 0 \cdot 4 + 5 \cdot 0 + 7 \cdot 8 = 2 + 0 + 0 + 56 = 58\,.$$

We *could* also compute $\mathbf{b}^T\mathbf{a}$:

$$\mathbf{b}^T\mathbf{a} = \begin{pmatrix} 2 & 4 & 0 & 8 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 5 \\ 7 \end{pmatrix} = 2 \cdot 1 + 4 \cdot 0 + 0 \cdot 5 + 8 \cdot 7 = 2 + 0 + 0 + 56 = 58$$

but, trivially, we will always get $\mathbf{a}^T\mathbf{b} = \mathbf{b}^T\mathbf{a}$ for two vectors of the same dimension. Thus,

$$\mathbf{a}^T\mathbf{b} = \mathbf{b}^T\mathbf{a} = 58\,.$$

Now, given that $\mathbf{a}^T\mathbf{b}$ and $\mathbf{b}^T\mathbf{a}$ are scalars, we immediately get

$$\mathbf{a}^T\mathbf{b}\mathbf{a} = 58\mathbf{a} = \begin{pmatrix} 58 \cdot 1 \\ 58 \cdot 0 \\ 58 \cdot 5 \\ 58 \cdot 7 \end{pmatrix} = \begin{pmatrix} 58 \cdot 1 \\ 58 \\ 0 \\ 290 \\ 406 \end{pmatrix}$$

and

$$\mathbf{b}^T\mathbf{a}\mathbf{b} = 58\mathbf{b} = \begin{pmatrix} 58 \cdot 2 \\ 58 \cdot 4 \\ 58 \cdot 0 \\ 58 \cdot 8 \end{pmatrix} = \begin{pmatrix} 58 \cdot 1 \\ 116 \\ 232 \\ 0 \\ 464 \end{pmatrix}\,.$$

(ii) To compute the outer product $\mathbf{a}\mathbf{b}^T$, we perform the following steps:

$$\mathbf{a}\mathbf{b}^T = \begin{pmatrix} 1 \\ 0 \\ 5 \\ 7 \end{pmatrix} \begin{pmatrix} 2 & 4 & 0 & 8 \end{pmatrix} = \begin{pmatrix} 1 \cdot 2 & 1 \cdot 4 & 1 \cdot 0 & 1 \cdot 8 \\ 0 \cdot 2 & 0 \cdot 4 & 0 \cdot 0 & 0 \cdot 8 \\ 5 \cdot 2 & 5 \cdot 4 & 5 \cdot 0 & 5 \cdot 8 \\ 7 \cdot 2 & 7 \cdot 4 & 7 \cdot 0 & 7 \cdot 8 \end{pmatrix} = \begin{pmatrix} 2 & 4 & 0 & 8 \\ 0 & 0 & 0 & 0 \\ 10 & 20 & 0 & 40 \\ 14 & 28 & 0 & 56 \end{pmatrix}$$

Similarly, to compute the outer product $\mathbf{b}\mathbf{a}^T$, we perform the following steps:

$$\mathbf{b}\mathbf{a}^T = \begin{pmatrix}2\\4\\0\\8\end{pmatrix}\begin{pmatrix}1 & 0 & 5 & 7\end{pmatrix} = \begin{pmatrix}2\cdot 1 & 2\cdot 0 & 2\cdot 5 & 2\cdot 7\\ 4\cdot 1 & 4\cdot 0 & 4\cdot 5 & 4\cdot 7\\ 0\cdot 1 & 0\cdot 0 & 0\cdot 5 & 0\cdot 7\\ 8\cdot 1 & 8\cdot 0 & 8\cdot 5 & 8\cdot 7\end{pmatrix} = \begin{pmatrix}2 & 0 & 10 & 14\\ 4 & 0 & 20 & 28\\ 0 & 0 & 0 & 0\\ 8 & 0 & 40 & 56\end{pmatrix}$$

Hopefully you noticed that these two matrices are the tranpose of each other! Of course that makes sense, because

$$(\boldsymbol{a}\boldsymbol{b}^\top)^\top = (\boldsymbol{b}^\top)^\top\boldsymbol{a}^\top = \boldsymbol{b}\boldsymbol{a}^\top .$$

2.2. Given the following matrix A and vector $\mathbf{b}$:

$$A = \begin{pmatrix}1 & 2 & 3\\ 4 & 5 & 6\\ 7 & 8 & 9\end{pmatrix} \quad \& \quad \mathbf{b} = \begin{pmatrix}1\\0\\-1\end{pmatrix}$$

compute the two products $A\mathbf{b}$ and $\mathbf{b}^\top A$.

Solution:

$$A\mathbf{b} = \begin{pmatrix}1 & 2 & 3\\ 4 & 5 & 6\\ 7 & 8 & 9\end{pmatrix}\begin{pmatrix}1\\0\\-1\end{pmatrix} = \begin{pmatrix}1\cdot 1 + 2\cdot 0 + 3\cdot(-1)\\ 4\cdot 1 + 5\cdot 0 + 6\cdot(-1)\\ 7\cdot 1 + 8\cdot 0 + 9\cdot(-1)\end{pmatrix} = \begin{pmatrix}1+0-3\\ 4+0-6\\ 7+0-9\end{pmatrix} = \begin{pmatrix}-2\\-2\\-2\end{pmatrix}$$

Furthermore,

$$\begin{aligned}\mathbf{b}^T A &= \begin{pmatrix}1 & 0 & -1\end{pmatrix}\begin{pmatrix}1 & 2 & 3\\ 4 & 5 & 6\\ 7 & 8 & 9\end{pmatrix}\\ &= \begin{pmatrix}[1\cdot 1 + 0\cdot 4 + (-1)\cdot 7] & [1\cdot 2 + 0\cdot 5 + (-1)\cdot 8] & [1\cdot 3 + 0\cdot 6 + (-1)\cdot 9]\end{pmatrix}\\ &= \begin{pmatrix}-6 & -6 & -6\end{pmatrix}\end{aligned}$$

Note: If we write $v = \begin{pmatrix}-6\\-6\\-6\end{pmatrix}$*, then the above solution is equal to* $v^\top \in \mathbb{R}^{1\times 3}$.

2.3. Given the matrices

$$\mathbf{A} = \begin{pmatrix}1 & 0 & 3\\ 0 & 5 & 0\end{pmatrix} \quad \& \quad \mathbf{B} = \begin{pmatrix}0 & 2\\ 4 & 0\\ 0 & 6\end{pmatrix}$$

compute the product $\mathbf{AB}$.

Solution:

$$\mathbf{C} = \mathbf{AB} = \begin{pmatrix}1 & 0 & 3\\ 0 & 5 & 0\end{pmatrix}\begin{pmatrix}0 & 2\\ 4 & 0\\ 0 & 6\end{pmatrix}$$

Calculating the elements of $\mathbf{C}$:

$$\begin{aligned}c_{11} &= 1\cdot 0 + 0\cdot 4 + 3\cdot 0 = 0\\ c_{12} &= 1\cdot 2 + 0\cdot 0 + 3\cdot 6 = 2 + 18 = 20\\ c_{21} &= 0\cdot 0 + 5\cdot 4 + 0\cdot 0 = 20\\ c_{22} &= 0\cdot 2 + 5\cdot 0 + 0\cdot 6 = 0\end{aligned}$$

Thus, the resulting matrix is:

$$\mathbf{AB} = \begin{pmatrix} 0 & 20 \\ 20 & 0 \end{pmatrix}$$

3. Question: Verification of Matrix Properties (*basic*)

3.1. Verify the **Associativity** and **Distributivity** properties of real-valued matrix operations (i.e. show that they are true). (*elementary*)

Solution:
Note: Hereafter, $\mathbf{M}_{i*}$ denotes the ith row of matrix $\mathbf{M}$ and $\mathbf{M}_{*j}$ its jth column.

To prove the left-hand distributive property, demonstrate the corresponding entries in the matrices $\mathbf{A}(\mathbf{B}+\mathbf{C})$ and $\mathbf{AB}+\mathbf{AC}$ are equal. To this end, use the definition of matrix multiplication to write

$$\begin{aligned}
[\mathbf{A}(\mathbf{B}+\mathbf{C})]_{ij} = \mathbf{A}_{i*}(\mathbf{B}+\mathbf{C})_{*j} = \sum_k [\mathbf{A}]_{ik}[\mathbf{B}+\mathbf{C}]_{kj} &= \sum_k [\mathbf{A}]_{ik}\left([\mathbf{B}]_{kj} + [\mathbf{C}]_{kj}\right) \\
&= \sum_k \left([\mathbf{A}]_{ik}[\mathbf{B}]_{kj} + [\mathbf{A}]_{ik}[\mathbf{C}]_{kj}\right) = \sum_k [\mathbf{A}]_{ik}[\mathbf{B}]_{kj} + \sum_k [\mathbf{A}]_{ik}[\mathbf{C}]_{kj} \\
&= \mathbf{A}_{i*}\mathbf{B}_{*j} + \mathbf{A}_{i*}\mathbf{C}_{*j} = [\mathbf{AB}]_{ij} + [\mathbf{AC}]_{ij} \\
&= [\mathbf{AB}+\mathbf{AC}]_{ij}
\end{aligned}$$

Since this is true for each i and j, it follows that $\mathbf{A}(\mathbf{B}+\mathbf{C}) = \mathbf{AB}+\mathbf{AC}$. The proof of the right-hand distributive property works analogously.
To prove the associative law, suppose that $\mathbf{B}$ is $p \times q$ and $\mathbf{C}$ is $q \times n$, and recall that the j^{th} column of $\mathbf{BC}$ is a linear combination of the columns in B. That is,

$$[\mathbf{BC}]_{*j} = \mathbf{B}_{*1}c_{1j} + \mathbf{B}_{*2}c_{2j} + \cdots + \mathbf{B}_{*q}c_{qj} = \sum_{k=1}^{q} \mathbf{B}_{*k}c_{kj}$$

Use this along with the left-hand distributive property to write

$$\begin{aligned}
[\mathbf{A}(\mathbf{BC})]_{ij} = \mathbf{A}_{i*}[\mathbf{BC}]_{*j} = \mathbf{A}_{i*}\sum_{k=1}^{q} \mathbf{B}_{*k}c_{kj} &= \sum_{k=1}^{q} \mathbf{A}_{i*}\mathbf{B}_{*k}c_{kj} \\
&= \sum_{k=1}^{q} [\mathbf{AB}]_{ik}c_{kj} = [\mathbf{AB}]_{i*}\mathbf{C}_{*j} = [(\mathbf{AB})\mathbf{C}]_{ij}\,.
\end{aligned}$$

3.2. Show that Associativity and Distributivity also hold for multiplying a scalar with real-valued matrixes, i.e. that the following statements are true $\forall \lambda, \psi \in \mathbb{R}$:

(i) Associativity:

$$(\lambda\psi)\boldsymbol{C} = \lambda(\psi\boldsymbol{C}), \quad \boldsymbol{C} \in \mathbb{R}^{m\times n}$$
$$\text{and } \lambda(\boldsymbol{BC}) = (\lambda\boldsymbol{B})\boldsymbol{C} = \boldsymbol{B}(\lambda\boldsymbol{C}) = (\boldsymbol{BC})\lambda, \quad \boldsymbol{B} \in \mathbb{R}^{m\times n}, \boldsymbol{C} \in \mathbb{R}^{n\times k}.$$

(ii) Distributivity:

$$\begin{aligned}
(\lambda+\psi)\boldsymbol{C} &= \lambda\boldsymbol{C} + \psi\boldsymbol{C}, & \boldsymbol{C} \in \mathbb{R}^{m\times n} \\
\text{and } \lambda(\boldsymbol{B}+\boldsymbol{C}) &= \lambda\boldsymbol{B} + \lambda\boldsymbol{C}, & \boldsymbol{B}, \boldsymbol{C} \in \mathbb{R}^{m\times n}
\end{aligned}$$

(*elementary*)

Solution:

(i) The first statement follows directly from how scalar multiplication is defined for matrices:

$$\forall i \in \{1, \ldots, m\},\ j \in \{1, \ldots, n\}: \quad [(\lambda\psi)\boldsymbol{C}]_{ij} = (\lambda\psi)c_{ij} = \lambda(\psi c_{ij}) = [\lambda(\psi\boldsymbol{C})]_{ij}$$

Furthermore, $\forall i \in \{1, \ldots, m\},\ j = \{1, \ldots, k\}:$

$$\begin{aligned}[\lambda(\boldsymbol{BC})]_{ij} = \lambda\sum_{l=1}^{n} b_{il}c_{lj} &= \sum_{l=1}^{n} \lambda b_{il}c_{lj} = \sum_{l=1}^{n}(\lambda b_{il})c_{lj} = [(\lambda\boldsymbol{B})\boldsymbol{C}]_{ij} \\ &= \sum_{l=1}^{n} b_{il}\lambda c_{lj} = \sum_{l=1}^{n} b_{il}(\lambda c_{lj}) = [\boldsymbol{B}(\lambda\boldsymbol{C})]_{ij} \\ &= \sum_{l=1}^{n} b_{il}c_{lj}\lambda = \sum_{l=1}^{n}(b_{il}c_{lj})\lambda = [(\boldsymbol{BC})\lambda]_{ij}\end{aligned}$$

(ii) The first statement again follows directly from how scalar multiplication is defined for matrices:

$$\forall i \in \{1, \ldots, m\},\ j \in \{1, \ldots, n\}: \quad [(\lambda + \psi)\boldsymbol{C}]_{ij} = (\lambda + \psi)c_{ij} = \lambda c_{ij} + \psi c_{ij} = [\lambda\boldsymbol{C} + \psi\boldsymbol{C}]_{ij}$$

Furthermore,

$$\forall i \in \{1, \ldots, m\},\ j \in \{1, \ldots, n\}: \quad [\lambda(\boldsymbol{B} + \boldsymbol{C})]_{ij} = \lambda(b_{ij} + c_{ij}) = \lambda b_{ij} + \lambda c_{ij} = [\lambda\boldsymbol{B} + \lambda\boldsymbol{C}]_{ij}$$

3.3. Prove that if a square matrix $\boldsymbol{A}$ is invertible, its inverse $\boldsymbol{A}^{-1}$ is unique.
Hint: *Generally, one can prove uniqueness by considering two elements that satisfy the given property (here, being the inverse of* $\boldsymbol{A}$*) and show that the two elements are equal to each other.*
(***slightly less elementary***)

Solution:

Suppose B and C are two inverses of A. We need to prove $B = C$. We have

$$AB = BA = I \quad \text{and} \quad AC = CA = I.$$

Therefore,

$$B = BI = B(AC) = (BA)C = IC = C.$$

So, the proof is complete.

3.4. Prove that the following holds for two non-zero $n \times n$, $n \in \mathbb{N}_{>0}$ matrices A, B and **invertible** $n \times n$ matrix C

$$AC = BC \Longrightarrow A = B \quad \text{and} \quad CA = CB \Longrightarrow A = B\,.$$

(***less elementary***)

Solution:

Suppose $AC = BC$. Multiply this equation by C^{-1} from the right side (we can do this because it is given that C has an inverse), we get

$$(AC)C^{-1} = (BC)C^{-1}. \text{ So} \quad A\left(CC^{-1}\right) = B\left(CC^{-1}\right) \quad \text{So} \quad AI = BI.$$

So $A = B$. Similarly, we prove the other one. The proof is complete.
Note: This is also called the ***Cancellation Property****, which only applies to invertible matrices, see also question 1.3.*

4. Question: Systems of Linear Equations (*basic*)

4.1. Find all solutions of the following system of linear equations. (*elementary*)

$$\begin{aligned} 4x_2 + 8x_3 &= 12 \\ x_1 - x_2 + 3x_3 &= -1 \\ 3x_1 - 2x_2 + 5x_3 &= 6 \end{aligned}$$

Solution:

First, let's write the corresponding augmented matrix.

$$\left[\begin{array}{ccc|c} 0 & 4 & 8 & 12 \\ 1 & -1 & 3 & -1 \\ 3 & -2 & 5 & 6 \end{array}\right]$$

Now we can use Guassian elimination

$$\left[\begin{array}{ccc|c} 0 & 4 & 8 & 12 \\ 1 & -1 & 3 & -1 \\ 3 & -2 & 5 & 6 \end{array}\right] \xrightarrow{\text{Switch } R_1 \text{ and } R_2} \left[\begin{array}{ccc|c} 1 & -1 & 3 & -1 \\ 0 & 4 & 8 & 12 \\ 3 & -2 & 5 & 6 \end{array}\right] \xrightarrow{R_3 - 3R_1 \to R_3} \left[\begin{array}{ccc|c} 1 & -1 & 3 & -1 \\ 0 & 4 & 8 & 12 \\ 0 & 1 & -4 & 9 \end{array}\right]$$

$$\xrightarrow{\frac{1}{4}R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & -1 & 3 & -1 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & -4 & 9 \end{array}\right] \xrightarrow{R_3 - R_2 \to R_3} \left[\begin{array}{ccc|c} 1 & -1 & 3 & -1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & -6 & 6 \end{array}\right]$$

$$\xrightarrow{-\frac{1}{6}R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & -1 & 3 & -1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

Now we use back-substitution to find the solution. The system of equations corresponding to the last augmented matrix above is:

$$\begin{aligned} x_1 - x_2 + 3x_3 &= -1 \\ x_2 + 2x_3 &= 3 \\ x_3 &= -1 \end{aligned}$$

So we know $x_3 = -1$. Plugging this into the second equation, we see that $x_2 = 3 - 2(-1) = 5$. Plugging both of these into the first equation, we get $x_1 = -1 + x_2 - 3x_3 = -1 + 5 - 3(-1) = 7$.

So the final answer is $x_1 = 7, x_2 = 5, x_3 = -1$.

4.2. Consider the following system of linear equations:

$$\begin{array}{rcr} -2x_1 + 4x_2 - 2x_3 - x_4 + 4x_5 &=& -3 \\ 4x_1 - 8x_2 + 3x_3 - 3x_4 + x_5 &=& 2 \\ x_1 - 2x_2 + x_3 - x_4 + x_5 &=& 0 \\ x_1 - 2x_2 - 3x_4 + 4x_5 &=& a \end{array} .$$

For which $a \in \mathbb{R}$ can it be solved? Give **one** particular solution to this linear system. (*slightly less elementary*)

Solution:

We start with the augmented matrix (in the form $[\boldsymbol{A} \mid \boldsymbol{b}]$)

$$\left[\begin{array}{rrrrr|r} -2 & 4 & -2 & -1 & 4 & -3 \\ 4 & -8 & 3 & -3 & 1 & 2 \\ 1 & -2 & 1 & -1 & 1 & 0 \\ 1 & -2 & 0 & -3 & 4 & a \end{array}\right]$$

Swapping Rows 1 and 3 leads to

$$\left[\begin{array}{ccccc|c} 1 & -2 & 1 & -1 & 1 & 0 \\ 4 & -8 & 3 & -3 & 1 & 2 \\ -2 & 4 & -2 & -1 & 4 & -3 \\ 1 & -2 & 0 & -3 & 4 & a \end{array}\right]$$

$$\xrightarrow{R_2-4R_1 \& R_3+2R_1 \& R_4-R_1} \left[\begin{array}{ccccc|c} 1 & -2 & 1 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 & -3 & 2 \\ 0 & 0 & 0 & -3 & 6 & -3 \\ 0 & 0 & -1 & -2 & 3 & a \end{array}\right]$$

$$\xrightarrow{R_4-R_2-R_3} \left[\begin{array}{ccccc|c} 1 & -2 & 1 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 & -3 & 2 \\ 0 & 0 & 0 & -3 & 6 & -3 \\ 0 & 0 & 0 & 0 & 0 & a+1 \end{array}\right]$$

$$\xrightarrow{R_2\cdot(-1) \& R_3\cdot(-\frac{1}{3})} \left[\begin{array}{ccccc|c} 1 & -2 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 3 & -2 \\ 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & a+1 \end{array}\right]$$

Now we have reached reduced row echelon form. Reverting this compact notation back into the explicit notation with the variables we seek, we obtain

$$\begin{aligned} x_1 - 2x_2 + x_3 - x_4 + x_5 &= 0 \\ x_3 - x_4 + 3x_5 &= -2 \\ x_4 - 2x_5 &= 1 \\ 0 &= a+1 \end{aligned}$$

Clearly, this is only solvable for $a = -1$.

Furthermore, one particular solution is given by

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}.$$

4.3. Let $\mathbf{s}_p \in \mathbb{R}^n$ be a (particular) solution to a system of linear equations defined by $\mathbf{Ax} = \mathbf{b}$, with $\mathbf{A} \in \mathbb{R}^{n\times n}$, $\mathbf{b} \in \mathbb{R}^n$. Furthermore, consider the set

$$S = \{\mathbf{s} \in \mathbb{R}^n \mid \mathbf{s} \text{ is a solution to the linear system } \mathbf{Ax} = \mathbf{0}_n\} .$$

Prove that $\forall \mathbf{s} \in S$, $\mathbf{s}_p + \mathbf{s}$ is a solution to $\mathbf{Ax} = \mathbf{b}$.
What are the possible sizes of the sets of all possible solutions for any given linear system? (***slightly more challenging***)

Solution:
We have, $\forall \mathbf{s} \in S$:

$$\mathbf{A}(\mathbf{s}_p + \mathbf{s}) = \mathbf{A}\mathbf{s}_p + \mathbf{A}\mathbf{s} = \mathbf{b} + \mathbf{0} = \mathbf{b}\,,$$

which proves the statement.
Generally, a system of linear equations has either *no solution* (size 0), or *exactly one solution* (size 1), or *infinitely many solutions* (size ∞).
This immediately becomes apparent by considering Illustration 2.1 in the booklet.

5. Question: Inverse of a matrix *(slightly more challenging)*

5.1. To compute the inverse $\boldsymbol{A}^{-1}$ of $\boldsymbol{A} \in \mathbb{R}^{n \times n}$, we need to find a matrix $\boldsymbol{A}^{-1}$ that satisfies $\boldsymbol{A}\boldsymbol{A}^{-1} = \boldsymbol{I}_n$. We can do this by transforming $[\boldsymbol{A} \mid \boldsymbol{I}_n]$ to $\left[\boldsymbol{I}_n \mid \boldsymbol{A}^{-1}\right]$, specifically by applying Gaussian elimination until the left side is the identity matrix, in which case the right side give the inverse.

Apply this principle to calculate the inverse of the following matrix:

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

Solution:
We write down the augmented matrix

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 2 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 2 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

and use Gaussian elimination to bring it into reduced row-echelon form

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -1 & 2 & -2 & 2 \\ 0 & 1 & 0 & 0 & 1 & -1 & 2 & -2 \\ 0 & 0 & 1 & 0 & 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 1 & -1 & 0 & -1 & 2 \end{array}\right],$$

such that the desired inverse is given as its right-hand side:

$$\boldsymbol{A}^{-1} = \begin{bmatrix} -1 & 2 & -2 & 2 \\ 1 & -1 & 2 & -2 \\ 1 & -1 & 1 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix}$$

You can verify this is indeed the inverse by performing the multiplication $\boldsymbol{A}\boldsymbol{A}^{-1}$ and observing that we recover $\boldsymbol{I}_4$.

5.2. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a non-zero 2×2 matrix. Show that

(1.) If $ad - bc = 0$, then A has no inverse and

(2.) If $ad - bc \neq 0$, then

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Solution:
Write

$$B = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

First,

$$AB = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} ad - bc & 0 \\ 0 & ad - bc \end{bmatrix} = (ad - bc)I_2.$$

1. (Case 1:) Assume $ad - bc = 0$. Then, we have $AB = O$. So, A cannot have an inverse (otherwise we will get $B = A^{-1}(AB) = O$, which is not the case).

2. (Case 2:) Assume $ad - bc \neq 0$. We need to prove $A\left(\frac{1}{ad-bc}B\right) = I_2 = \left(\frac{1}{ad-bc}B\right)A$. Multiply the above equation by $\frac{1}{ad-bc}$, we get

$$A\left(\frac{1}{ad-bc}B\right) = I_2$$

Similarly, $\left(\frac{1}{ad-bc}B\right)A = I_2$. So, the proof is complete.

6. Question: *Freaky Fun*

In math, computing, etc., the modulo operation gives us the remainder when one integer is divided by another integer. Specifically, for $a, n \in \mathbb{Z}$, we have

$$a \mod n := a - n\left\lfloor \frac{a}{n} \right\rfloor .$$

Now, consider the set $\mathbb{F}_7 = \{1, 2, 3, 4, 5, 6\}$ with the following two operations:

$$\begin{aligned} + :& \mathbb{F}_7 \times \mathbb{F}_7 \to \mathbb{F}_7, (x, y) \longmapsto x + y \text{ mod } 7 \\ \cdot :& \mathbb{F}_7 \times \mathbb{F}_7 \to \mathbb{F}_7, (x, y) \longmapsto xy \text{ mod } 7 \end{aligned}$$

(*In case this reminds you of quotient spaces you're not wrong, but we won't even talk about vector spaces until next week.*)
Assuming that addition and multiplication of matrices with entries in $\mathbb{F}_7$ is defined analogously to the real-valued case, calculate $A + B$; AB; and BA for matrices

$$A = \begin{pmatrix} 0 & 2 & 4 \\ 1 & 5 & 6 \\ 0 & 1 & 0 \end{pmatrix} \quad \& \quad B = \begin{pmatrix} 1 & 1 & 0 \\ 5 & 3 & 1 \\ 0 & 4 & 6 \end{pmatrix} .$$

Solution:

- $$A + B = \begin{pmatrix} 0+1 & 2+1 & 4+0 \\ 1+5 & 5+3 & 6+1 \\ 0+0 & 1+4 & 0+6 \end{pmatrix} \overset{\text{applying mod 7 directly}}{=} \begin{pmatrix} 1 & 3 & 4 \\ 6 & 1 & 0 \\ 0 & 5 & 6 \end{pmatrix}$$

- Using regular real-valued multiplication and addition, we get

$$AB = \begin{pmatrix} 0 & 2 & 4 \\ 1 & 5 & 6 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 5 & 3 & 1 \\ 0 & 4 & 6 \end{pmatrix} = \begin{pmatrix} (0\cdot1+2\cdot5+4\cdot0) & (0\cdot1+2\cdot3+4\cdot4) & (0\cdot0+2\cdot1+4\cdot6) \\ (1\cdot1+5\cdot5+6\cdot0) & (1\cdot1+5\cdot3+6\cdot4) & (1\cdot0+5\cdot1+6\cdot6) \\ (0\cdot1+1\cdot5+0\cdot0) & (0\cdot1+1\cdot3+0\cdot4) & (0\cdot0+1\cdot1+0\cdot6) \end{pmatrix}$$

  Calculating the entries modulo 7, we get:

$$AB = \begin{pmatrix} 3 & 1 & 5 \\ 5 & 5 & 6 \\ 5 & 3 & 1 \end{pmatrix}$$

- Using regular real-valued multiplication and addition, we get

$$BA = \begin{pmatrix} 1 & 1 & 0 \\ 5 & 3 & 1 \\ 0 & 4 & 6 \end{pmatrix} \begin{pmatrix} 0 & 2 & 4 \\ 1 & 5 & 6 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} (1\cdot0+1\cdot1+0\cdot0) & (1\cdot2+1\cdot5+0\cdot1) & (1\cdot4+1\cdot6+0\cdot0) \\ (5\cdot0+3\cdot1+1\cdot0) & (5\cdot2+3\cdot5+1\cdot1) & (5\cdot4+3\cdot6+1\cdot0) \\ (0\cdot0+4\cdot1+6\cdot0) & (0\cdot2+4\cdot5+6\cdot1) & (0\cdot4+4\cdot6+6\cdot0) \end{pmatrix}$$

Calculating the entries modulo 7, we get:

$$BA = \begin{pmatrix} 1 & 0 & 3 \\ 3 & 5 & 3 \\ 4 & 5 & 3 \end{pmatrix}$$

If you have any questions or feedback, please feel free to contact me via E-mail at hannah.kuempel@stat.uni-muenchen.de!!

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