1. Question: Integration w.r.t. different measures (*elementary*)

1.1. Consider the function

$$f : \mathbb{R} \longrightarrow \mathbb{R}, \quad x \mapsto \begin{cases} 1, & \text{if } -1 < x \leq 0 \\ 2, & \text{if } 0 < x \leq 1 \\ 3, & \text{if } 1 < x \leq 2, \\ 0, & \text{otherwise.} \end{cases}$$

Calculate the integral of this function w.r.t.

(i) the Lebesgue measure and

(ii) the Dirac measure, defined on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ as $\delta_y(x) = \mathbb{1}_{x=y}$ for a fixed $y \in \mathbb{R}$.

1.2. Let μ be the counting measure on $\mathbb{N}$, and define the sequence $\{f_n\}_{n \in \mathbb{N}}$ by

$$f_n(x) = \begin{cases} 1 & \text{if } x = n \\ 0 & \text{otherwise.} \end{cases}$$

Compute

(i) $\lim_{n\to\infty} \int f_n d\mu$ and

(ii) $\int \lim_{n\to\infty} f_n d\mu$.

Solution:

1.1. (i) Since f is a simple and nonnegative function, we can simply write the integral as

$$\int f(x)\lambda(x) = 1 \cdot \lambda\left((-1, 0]\right) + 2 \cdot \lambda\left((0, 1]\right) + 3 \cdot \lambda\left((1, 2]\right) = 1 + 2 + 3 = 6.$$

(ii) For any simply function $g(x) = \sum_i c_i \mathbb{1}_{A_i}(x)$, the integral w.r.t. the Dirac measure is given by

$$\int g d\delta_y = \sum_i c_i \delta_y\left(A_i\right) = \sum_i c_i \mathbb{1}_{A_i}(y) = g(y).$$

Therefore, $\int f d\delta_y = f(y)$.

1.2. (i) $\lim_{n\to\infty} \int f_n d\mu = 1$

(ii) $\int \lim_{n\to\infty} f_n d\mu = 0$

This shows that the integral and limit are not always interchangeable.

2. Question: Measures and Probability Space (*elementary*)

2.1. Take the measurable space $\Omega = \{1, 2\}, F = 2^{\Omega}$. Which of the following is a measure? Which is a probability measure?

a. $\mu(\emptyset) = 0, \mu(\{1\}) = 5, \mu(\{2\}) = 6, \mu(\{1, 2\}) = 11$

b. $\mu(\emptyset) = 0, \mu(\{1\}) = 0, \mu(\{2\}) = 0, \mu(\{1, 2\}) = 1$

c. $\mu(\emptyset) = 0, \mu(\{1\}) = 0, \mu(\{2\}) = 0, \mu(\{1, 2\}) = 0$

d. $\mu(\emptyset) = 0, \mu(\{1\}) = 0, \mu(\{2\}) = 1, \mu(\{1, 2\}) = 1$

e. $\mu(\emptyset) = 0, \mu(\{1\}) = 0, \mu(\{2\}) = \infty, \mu(\{1, 2\}) = \infty$

2.2. Define a probability space that could be used to model the outcome of throwing two fair 6-sided dice.

2.3. Let (X, μ) be a measure space and E a measurable subset of X. Show that for every $A \subset X$ the following holds:

$$\mu(E \cap A) + \mu(E \cup A) = \mu(E) + \mu(A).$$

2.4. Let A and B be events with probabilities $P(A) = \frac{2}{3}$ and $P(B) = \frac{1}{2}$

(i) Show that $\frac{1}{6} \leq P(A \cap B) \leq \frac{1}{2}$, and give examples to show that both extremes are possible.

(ii) Find corresponding bounds for $P(A \cup B)$.

Solution:

2.1.
a. Measure. Not probability measure since $\mu(\Omega) > 1$.
b. Neither due to countable additivity.
c. Measure. Not probability measure since $\mu(\Omega) = 0$.
d. Probability measure.
e. Measure. Not probability measure since $\mu(\Omega) > 1$.

2.2.
- $\Omega = \{\{i, j\}, i = 1, \ldots, 6, j = 1, \ldots, 6\}$
- $F = 2^{\Omega}$
- $\forall \omega \in \Omega, P(\omega) = \frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$

2.3. Due to the measurability of E, we know

$$\mu(E \cup A) = \mu((E \cup A) \cap E) + \mu\left((E \cup A) \cap E^c\right) = \mu(E) + \mu\left(A \cap E^c\right)$$

and similarily

$$\mu(A) = \mu(A \cap E) + \mu\left(A \cap E^c\right)$$

Comparing the expressions for $\mu\left(A \cap E^c\right)$, we obtain

$$\mu(E \cap A) + \mu(E \cup A) = \mu(E) + \mu(A).$$

2.4. (i) From the properties of probability we have

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) \leq 1$$

From this follows

$$\begin{aligned} P(A \cap B) &\geq P(A) + P(B) - 1 \\ &= \frac{2}{3} + \frac{1}{2} - 1 \\ &= \frac{1}{6}, \end{aligned}$$

which is the lower bound for the intersection. Conversely, we have

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) \geq P(A)$$

From this follows

$$\begin{aligned} P(A \cap B) &\leq P(B) \\ &= \frac{1}{2}, \end{aligned}$$

which is the upper bound for the intersection. For an example take a fair die. To achieve the lower bound let $A = \{3, 4, 5, 6\}$ and $B = \{1, 2, 3\}$, then their intersection is $A \cap B = \{3\}$. To achieve the upper bound take $A = \{1, 2, 3, 4\}$ and \$$B = \{1, 2, 3\}$\$.

(ii) For the bounds of the union we will use the results from the first part. Again from the properties of probability we have

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ &\geq P(A) + P(B) - \frac{1}{2} \\ &= \frac{2}{3}. \end{aligned}$$

Conversely

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ &\leq P(A) + P(B) - \frac{1}{6} \\ &= 1. \end{aligned}$$

Therefore $\frac{2}{3} \leq P(A \cup B) \leq 1$.

3. Question: Sigma Algebras *(medium)*

3.1. Let A be a fixed subset of a set X. Determine the σ-algebra of subsets of X generated by $\{A\}$.

3.2. Let Ω be a non-empty set. Suppose that $\mathcal{F}_1$ and $\mathcal{F}_2$ are σ-algebras on Ω. Prove that $\mathcal{F}_1 \cap \mathcal{F}_2$ is also a σ-algebra on Ω.

3.3. Suppose that $\mathcal{F}_1$ and $\mathcal{F}_2$ are σ-algebras on Ω. Show by example that $\mathcal{F}_1 \cup \mathcal{F}_2$ may fail to be a σ-algebra.
Hint: You can consider two σ-algebras $\mathcal{F}_1$ and $\mathcal{F}_2$ on $\Omega := \{1, 2, 3\}$.

3.4. Let X be an uncountable set. (An uncountable set X is one that is not countable, i.e. there is no bijection between X and $\mathbb{N}$, meaning X has more elements than the natural numbers.)
Consider

$$\mathcal{S} = \{E \subset X : E \text{ or } E^c \text{ is at most countable } \}$$

and show that $\mathcal{S}$ is a σ-algebra and that $\mathcal{S}$ is generated by the one-point subsets of X.
Hint: It will help to apply the following identity: $\cup_{k=1}^{\infty} A_k^c \subset (\cap_{k=1}^{\infty} A_k)^c$.

Solution:

3.1. The σ-algebra generated by $\{A\}$ necessarily contains the following elements:

$$\emptyset, A, A^c, X$$

Due to the collection $\{\emptyset, A, A^c, X\}$ already being closed under taking complements and unions of sets, this is the σ-algebra generated by $\{A\}$.

3.2. We check the requirements for a σ-algebra:

- $\Omega \in \mathcal{F}_1 \cap \mathcal{F}_2$ because $\Omega \in \mathcal{F}_i$ for all $i \in \{1, 2\}$;
- if $A \in \mathcal{F}_1 \cap \mathcal{F}_2$, then $A \in \mathcal{F}_i$, and hence $A^c \in \mathcal{F}_i$ for all $i \in \{1, 2\}$. It follows that $A^c \in \mathcal{F}_1 \cap \mathcal{F}_2$;
- if $A_n \in \mathcal{F}_1 \cap \mathcal{F}_2, n \in \mathbb{N}$, then $A_n \in \mathcal{F}_i$ for all $i \in \{1, 2\}$. Hence, $\cup_{n \in \mathbb{N}} A_n \in \mathcal{F}_i$ for all $i \in \{1, 2\}$, and thus $\cup_{n \in \mathbb{N}} A_n \in \mathcal{F}_1 \cap \mathcal{F}_2$.

3.3. Let $\Omega := \{1, 2, 3\}$, and consider the σ-algebras

$$\mathcal{F}_1 := \sigma(\{\{1\}\}) = \{\emptyset, \{1, 2, 3\}, \{1\}, \{2, 3\}\} \text{ and } \mathcal{F}_2 := \sigma(\{\{2\}\}) = \{\emptyset, \{1, 2, 3\}, \{2\}, \{1, 3\}\}$$

It is straightforward to verify that the union $\mathcal{F}_1 \cup \mathcal{F}_2$ contains both $\{1\}$ and $\{2\}$, yet does not contain $\{1\} \cup \{2\} = \{1, 2\}$. Hence, $\mathcal{F}_1 \cup \mathcal{F}_2$ is not a σ-algebra.

3.4. Firstly, let us show that $\mathcal{S}$ is a σ-algebra. Clearly, $\emptyset$ and X belong to $\mathcal{S}$. Moreover, it is easy to see that $\mathcal{S}$ is closed under taking complements. Let therefore $\{A_k\} \subset \mathcal{S}$. If all sets $\{A_k\}$ are at most countable, then so is $\cup_{k=1}^{\infty} A_k$, implying that the union again belongs to $\mathcal{S}$. Otherwise, A_m is uncountable for some m, therefore A_m^c is at most countable. Due to the inclusion

$$\left(\bigcup_{k=1}^{\infty} A_k\right)^c \subset A_m^c$$

the complement of $\cup_{k=1}^{\infty} A_k$ is at most countable and therefore

$$\bigcup_{k=1}^{\infty} A_k \in \mathcal{S}$$

It remains to show that $\mathcal{S}$ is generated by the one-point subsets of X. By the definition of $\mathcal{S}$, all one-point subsets belong to $\mathcal{S}$. In addition, for every A in $\mathcal{S}$, either A or its complement can be expressed as a countable union of one-point subsets. Consequently, every element in $\mathcal{S}$ can be obtain from the one-point subsets using unions and complements.

4. Question: More Measure Theory *(medium)*

4.1. Show that the Lebesgue measure of rational numbers on $[0, 1]$ is 0.

4.2. Take the measure space $(\Omega_1 = (0, 1], \mathcal{B}((0, 1]), \lambda)$ (we know that this is a probability space on $(0, 1]$).

(i) Define a map (function) from Ω_1 to $\Omega_2 = \{1, 2, 3, 4, 5, 6\}$ such that the measure space $\left(\Omega_2, 2^{\Omega_2}, \lambda \circ f^{-1}\right)$ will be a discrete probability space with uniform probabilities $\left(P(\omega) = \frac{1}{6}, \forall \omega \in \Omega_2\right)$.

(ii) Is the map that you defined in (i) the only such map?

(iii) How would you in the same fashion define a map that would result in a probability space that can be interpreted as a coin toss with probability p of heads?

4.3. Let $\Omega_1 = (0, 1)$, let $\mathcal{F}_1$ be the Borel sets, and let $\mathbb{P}_1$ be the Lebesgue measure. Let $\Omega_2 = (0, 1)$ let $\mathcal{F}_2$ be the set of all subsets of $(0, 1)$, and let $\mathbb{P}_2$ be the counting measure. In particular, for every infinite (countable or uncountable) subset of $(0, 1), \mathbb{P}_2(A) = \infty$.

Define

$$f : \mathbb{R}^2 \longrightarrow \mathbb{R}, \quad (x, y) = \begin{cases} 1, & \text{if } x = y \\ 0, & \text{otherwise.} \end{cases}$$

Does Fubini's theorem apply here?

Solution:

4.1. There are a countable number of rational numbers. Therefore, we can write

$$\begin{aligned} \lambda(\mathbb{Q}) &= \lambda\left(\cup_{i=1}^{\infty} q_i\right) && \\ &= \sum_{i=1}^{\infty} \lambda\left(q_i\right) && \text{(countable additivity)} \\ &= \sum_{i=1}^{\infty} 0 && \text{(Lebesgue measure of a singleton)} \\ &= 0. && \end{aligned}$$

4.2. (i) In other words, we have to assign disjunct intervals of the same size to each element of Ω_2. Therefore

$$f(x) = \lceil 6x \rceil$$

(ii) No, we could for example rearrange the order in which the intervals are mapped to integers.

Additionally, we could have several disjoint intervals that mapped to the same integer, as long as the Lebesgue measure of their union would be $\frac{1}{6}$ and the function would remain injective.

(iii) We have $\Omega_3 = \{0, 1\}$, where zero represents heads and one represents tails. Then

$$f(x) = 0^{\mathbb{1}_A(x)}$$

where $A = \{y \in (0, 1] : y < p\}$.

4.3. Here,

$$\int_{\Omega_1} \int_{\Omega_2} f(x, y) d\mathbb{P}_2(y) d\mathbb{P}_1(x) = \int_{\Omega_1} 1 d\mathbb{P}_1(y) = 1$$

but

$$\int_{\Omega_2} \int_{\Omega_1} f(x, y) d\mathbb{P}_1(x) d\mathbb{P}_2(y) = \int_{\Omega_2} 0 d\mathbb{P}_2(y) = 0$$

In this case, the conditions of Fubini's theorem fail to hold: the measure on $(0, 1)$ is not σ-finite.

5. Question: Infinite Monkey Theorem (*Freaky Fun*)

Prove the following statement: *Consider an infinite string of letters $a_1 a_2 \cdots a_n \cdots$ produced from a finite alphabet (of, say, 26 letters) by picking each letter independently at random, and uniformly from the alphabet (so each letter gets picked with probability $\frac{1}{26}$). Fix a string S of length m from the same alphabet (which is the given "text"). Let E_j be the event that the substring $a_j a_{j+1} \cdots a_{j+m-1}$ is S. Then with probability 1, infinitely many of the E_j's occur.*

Solution:

Consider the sequence of events $(E_{mj+1})_{j=0}^{\infty}$. Observe that they are independent events: the event that $a_1 a_2 \cdots a_m$ is S is independent from the event that $a_{m+1} a_{m+2} \cdots a_{2m}$ is S, etc., since they belong to different "blocks" of the infinite string. Moreover, for every j, $P(E_{mj+1}) = \left(\frac{1}{26}\right)^m$. Therefore $\sum_{j=0}^{\infty} P(E_{mj+1}) = \sum_{j=0}^{\infty} \left(\frac{1}{26}\right)^m = \infty$. So by Part (ii) of the Borel-Cantelli lemma, the probability that infinitely many of the E_{mj+1} 's occur is 1 .

If you have any questions or feedback, please feel free to contact me via E-mail at hannah.kuempel@stat.uni-muenchen.de!!

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