1. Question: Metrics on $\mathbb{R}$

1.1. What would be the natural norm and metric to define on the real line? Use this metric to show that the real line is a metric space.

Solution:
Clearly, the absolute value is a norm on $\mathbb{R}$. Therefore, by Remark 1.1., the function

$$d : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}_{\geq 0},\ (x, y) \mapsto |x - y|$$

is a metric on $\mathbb{R}$. Therefore, the real line $(\mathbb{R}, d)$ is a metric space.

1.2. Are the following functions metrics on $\mathbb{R}$? Prove your answer.

(i) $d : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}, \quad (x, y) \mapsto (x - y)^2$
(ii) $d : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}, \quad (x, y) \mapsto \sqrt{|x - y|}$.

Solution:

(i) No, this is not a metric, since it does not satisfy the triangle inequality. We can prove this via counterexample (or contradiction): Choose $x = 3, y = 1$ and $z = 2$, then

$$d(3, 1) = (3 - 1)^2 = 2^2 = 4$$

but

$$d(3, 2) + d(2, 1) = (3 - 2)^2 + (2 - 1)^2 = 2 < 4.$$

(ii) The first two properties of a metric are clearly fulfilled by $d(x, y) = \sqrt{|x - y|}$:

1. Consider $x, y \in \mathbb{R}$ with $x = y$. Then $d(x, y) = \sqrt{0} = 0$.
   Meanwhile, for $x, y \in \mathbb{R}$ with $x \neq y$, it follows that $q := |x - y| \in \mathbb{R}_{>0}$
   $\Rightarrow d(x, y) = \sqrt{q} > 0$.
2. Symmetry also directly follows from the symmetry of the absolute value, i.e. the fact that $|x - y| = |y - x|$.

To verify the triangle inequality for any $x, y, z \in \mathbb{R}$, we write

$$\begin{aligned}
[d(x, y)]^2 = |x - y| &\leq |x - z| + |z - y| \\
&\leq |x - z| + |z - y| + 2\sqrt{|x - z|}\sqrt{|z - y|} \\
&= (\sqrt{|x - z|} + \sqrt{|z - y|})^2 \\
&= \big[d(x, z) + d(z, y)\big]^2.
\end{aligned}$$

Taking square root on both sides yields the triangle inequality.

1.3. Let d be a metric on X. Determine all constants $k \in \mathbb{R}$ such that each of the following functions $d' : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ is a metric on X.

(i) $d'(x, y) := kd(x, y)$
(ii) $d'(x, y) := d(x, y) + k$.

Solution:

(i) Here, the answer is any $k \in \mathbb{R}_{>0}$. This is because, multiplication generally does not affect the properties of a metric. But, if X has more than one point (as $\mathbb{R}$ does), then the zero function cannot be a metric on X.

(ii) Here, k has to be equal to zero so that d' still satisfies the first requirement of a metric, i.e.

$\forall x, y \in \mathcal{S}$ we have $d(x,y) = 0$ if and only if $x = y$. This is because, if $k \neq 0$, it would follow that

$$d'(x,y) = d(x,y) + k = 0 + k \neq 0.$$

2. Question: Metrics on $\mathbb{R}^m$

2.1. Sketch the unit-ball, i.e. the ε-Ball in $\mathbb{R}^2$ with $\varepsilon = 1$, for the following three metrics:

- Euclidean distance, i.e.

$$d_{\text{Euclidean}} : \mathbb{R}^m \times \mathbb{R}^m \longrightarrow \mathbb{R}_{\geq 0}, \quad (\boldsymbol{x}, \boldsymbol{y}) \mapsto \sqrt{\sum_{i=1}^{m} (x_i - y_i)^2}$$

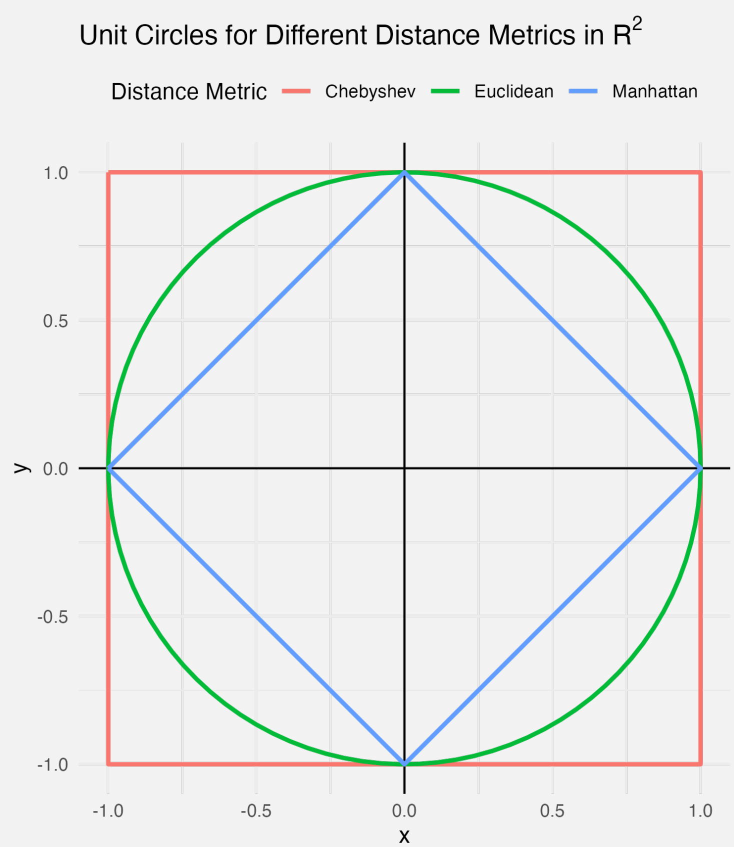
- Manhattan distance, i.e.

$$d_{\text{Manhattan}} : \mathbb{R}^m \times \mathbb{R}^m \longrightarrow \mathbb{R}_{\geq 0}, \quad (\boldsymbol{x}, \boldsymbol{y}) \mapsto \sum_{i=1}^{m} |x_i - y_i|$$

- Chebyshev distance, i.e.

$$d_{\text{Chebyshev}} : \mathbb{R}^m \times \mathbb{R}^m \longrightarrow \mathbb{R}_{\geq 0}, \quad (\boldsymbol{x}, \boldsymbol{y}) \mapsto \max\left\{|x_1 - y_1|, |x_2 - y_2|, \dots, |x_m - y_m|\right\}.$$

Solution:



2.2. Use the sketch from **2.1** to explain the intuition of each metric and give an example of when it might be useful.

Solution:

- **Euclidean Distance:** This is the most common distance metric. It represents our intuitive understanding of distance as the length of a straight line between two points.
- **Manhattan Distance:** As this metric sums up element-wise absolute distances, it may be thought of as the length of a step-wise function in $\mathbb{R}^2$. As such, it the Manhattan distance is, e.g., useful in grid-based environments, such as urban planning where movement is restricted to grid-like street layouts. Of course there are many more applications, like L1 regularization.
- **Chebyshev Distance:** As this metric gives us the maximum element-wise distance, it is helpful in scenarios where the maximum distance in any dimension is critical, such as in logistics where the worst-case scenario must be minimized.

2.3. is the squared Euclidean distance $d^2_{\text{Euclidean}}$ a metric? Prove your answer.

Solution:

No, it is not. Given 1.2(i), it is intuitive that $d_{\text{Euclidean}}$ would not satisfy the triangle inequality. To prove this, we generalize the contradiction/counterexample of 1.2(i):

Consider an arbitrary $x \in \mathbb{R}^m \backslash \{0\}$ and set $y = 3x$ and $z = 2x$. Then

$$d(x,y)^2 = \sum_{i=1}^m (x_i - 3x_i)^2 = 4\sum_{i=1}^m x_i^2$$

and

$$d(x,z)^2 + d(z,y)^2 = \sum_{i=1}^m x_i^2 + \sum_{i=1}^m x_i^2 = 2\sum_{i=1}^m x_i^2 \,.$$

Since $4\sum_{i=1}^m x_i^2 > 2\sum_{i=1}^m x_i^2$, we have found a contradiction/counterexample in which the triangle inequality does not apply. Therefore, the squared Euclidean distance isn't a metric.

3. Question: Triangle inequality

3.1. Prove the generalized triangle inequality, i.e. that for some metric space (X,d), $n > 2$, and $x_1, \ldots, x_{n-1}, x_n \in X$, it holds that

$$d(x_1, x_n) \leq d(x_1, x_2) + d(x_2, x_3) + \ldots + d(x_{n-1}, x_n)\ .$$

Solution:

We prove the generalized triangle inequality by induction. The case $n = 3$ follows from definition of a metric. Suppose the statement is true for $n = k$. For $n = k + 1$,

$$\begin{aligned} d(x_1, x_{k+1}) &\leq d(x_1, x_k) + d(x_k, x_{k+1}) \\ &\leq d(x_1, x_2) + d(x_2, x_3) + \ldots + d(x_{k-1}, x_k) + d(x_k, x_{k+1}) \end{aligned}$$

where the last inequality follows from the induction hypothesis. Since $k \geq 3$ is arbitrary, the statement follows from induction.

3.2. Using the triangle inequality, show that for any metric d

$$|d(x,z) - d(y,z)| \leq d(x,y)\,.$$

Solution:

Suppose (X, d) is a metric space. For any x, y, z in X, we have

$$d(x, z) \leq d(x, y) + d(y, z)$$

by the triangle inequality, so that

$$d(x, z) - d(y, z) \leq d(x, y)$$

Now interchange x and y. Using the symmetry of the metric d (property 2 of metrics) we also obtain

$$d(y, z) - d(x, z) \leq d(x, y)$$

Combining the last two inequalities we find

$$|d(x, z) - d(y, z)| \leq d(x, y)\,.$$

3.3. Using the triangle inequality, show that for any metric d

$$|d(x, y) - d(z, w)| \leq d(x, z) + d(y, w)\,.$$

Solution:

Suppose (X, d) is a metric space. For any x, y, z, w in X, the generalised triangle inequality yields

$$\begin{aligned} d(x, y) &\leq d(x, z) + d(z, w) + d(w, y) \\ \Longrightarrow d(x, y) - d(z, w) &\leq d(x, z) + d(w, y) \end{aligned}$$

Furthermore,

$$\begin{aligned} &= d(x, z) + d(y, w) && [\text{ by symmetry of } d\ ] \\ d(z, w) &\leq d(z, x) + d(x, y) + d(y, w) \\ \Longrightarrow d(z, w) - d(x, y) &\leq d(z, x) + d(y, w) \\ &= d(x, z) + d(y, w) && [\text{ by by symmetry of } d\ ] \end{aligned}$$

Again, combining these two inequalities yields the desired statement.

4. Question: Open sets

4.1. Are the following sets open or closed in the metric spaces $(\mathbb{R}^2, d_{\text{Euclidean}})$ and $(\mathbb{R}, d_{\text{Euclidean}})$, respectively? Prove your answer.

(a) $A = \left\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 2y\right\}$

(b) $B = \left\{x \in \mathbb{R} : x^3 + 2x^2 - 3x \leq 0\right\}$

Solution:

(a) A is an open set. Completing the square, one may express the given set in the form

$$\begin{aligned} A &= \left\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 - 2y < 0\right\} \\ &= \left\{(x, y) \in \mathbb{R}^2 : x^2 + (y - 1)^2 < 1\right\}, \end{aligned}$$

which, for $a = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is equal to the open ball $B_1(a)$.
In particular, A is open in $\mathbb{R}^2$ because every open ball is open in $\mathbb{R}^2$.

(b) B is a closed set. To prove this, it suffices to show that the complement of A is open in $\mathbb{R}$. Since

$$x^3 + 2x^2 - 3x = x\left(x^2 + 2x - 3\right) = x(x+3)(x-1)$$

one has $x^3 + 2x^2 - 3x > 0$ if and only if $x \in (-3, 0) \cup (1, \infty)$. Thus, the complement of B is the union of two open sets, so the complement of B is open and B is closed.

Note: Parentheses, (), denote open interval endpoints wile square brackets, [], denote closed endpoints. For example, the half-open interval $[0, 1)$ contains 0, but not 1.

4.2. Suppose (X, d) is a metric space and $f : X \to \mathbb{R}$ is continuous. Show that $B = \{x \in X : |f(x)| < r\}$ is open in X for each $r > 0$.

Solution:
The given set can be expressed in the form

$$A = \{x \in X : -r < f(x) < r\} = f^{-1}\big((-r, r)\big)\,.$$

Since $(-r, r)$ is open in $\mathbb{R}$, its inverse image A under continuous function f must then be open in X by Proposition 1.4.

4.3. Show that every function $f : X \to Y$ is continuous when X, Y are metric spaces and the metric on X is the so-called discrete metric, defined as

$$d : X \times X \longrightarrow \{0, 1\},\ (x, y) \mapsto \mathbb{1}_{x \neq y} := \begin{cases} 1, & x \neq y, \\ 0, & \text{otherwise, i.e. } x = y \end{cases}.$$

Solution:
We use the epsilon-delta criterion to show that any such f is continuous. Let $\varepsilon > 0$ be given and take $\delta = 1$. Then, $\forall x, y \in X$

$$d_X(x, y) < \delta \quad \Longrightarrow \quad x = y \quad \Longrightarrow \quad d_Y(f(x), f(y)) = d_Y(f(x), f(x)) = 0 < \varepsilon\,.$$

Since this holds $\forall x \in X$, every function $f : X \longrightarrow Y$ in this setting is continuous.

4.4. Suppose $f : X \to Y$ is a constant function between metric spaces, say $f(x) = y_0$ for all $x \in X$. Show that f is continuous.

Solution:
To show that f is continuous, we again use the epsilon-delta criterion.
Let $\varepsilon > 0$ be given and $\delta > 0$ be arbitrary. Then, $\forall x, y \in X$

$$d_X(x, y) < \delta \quad \Longrightarrow \quad d_Y(f(x), f(y)) = d_Y\left(y_0, y_0\right) = 0 < \varepsilon\,,$$

so f is continuous by the same reasoning as in 3.4.

If you have any questions or feedback, please feel free to contact me via E-mail at hannah.kuempel@stat.uni-muenchen.de!!

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