1. Question: Classifying critical points (*elementary*)

1.1. True or false? Motivate your answer. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a twice differentiable function with a critical point p_0, whose Hessian matrix at p_0 is

$$\boldsymbol{H} f\left(p_0\right)=\left(\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{array}\right).$$

Then:

(a) p_0 cannot be a local maximum
(b) p_0 cannot be a local minimum
(c) p_0 cannot be a saddle point
(d) none of the above.

1.2. True or false? Motivate your answers. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a continuously differentiable function and consider its restriction over the square $Q = [0, 1] \times [0, 1] \subset \mathbb{R}^2$. Then:

(a) If f has a local max / min / saddle at x_0 in Q, then $df\left(x_0\right) = 0$
(b) Let $x_0 \in Q$ be a point such that $df\left(x_0\right) = 0$, then f has a local max/min/saddle at x_0.

1.3. For each of the following functions, determine their critical points and find those for which the 2nd derivative test applies, determining in such case whether they are local maxima, local minima or saddle points.

(a) $f : \mathbb{R}^2 \to \mathbb{R}, \quad f(x, y) = x^3 + y^3 - 3xy,$
(b) $f : \mathbb{R}^3 \to \mathbb{R}, \quad f(x, y, z) = \left(x^3 - 3x - y^2\right) z^2 + z^3,$
(c) $f : \mathbb{R}^2 \to \mathbb{R}, \quad f(x, y) = xy^2 - \cos(x).$

2. Question: Second derivative test for $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ (*elementary*)

2.1. Prove the following statement. (***A bit more advanced, you may also just use this statement for the following question for now.***)

Let $A = \left(\begin{array}{cc} a & b \\ b & c \end{array}\right)$ *be a symmetric* 2×2 *matrix, where* $a, b, c \in \mathbb{R}$*. Then* A *is positive definite if* $a > 0$ *and* $ac - b^2 > 0$*.*A *is negative definite if* $a < 0$ *and* $ac - b^2 > 0$*.*

2.2. Show that if $f : A \subseteq \mathbb{R}^2 \to \mathbb{R}$ has a critical point $x_0 \in A$ and we let

$$\Delta = \frac{\partial^2 f}{\partial x_1 \partial x_1} \cdot \frac{\partial^2 f}{\partial x_2 \partial x_2} - \left(\frac{\partial^2 f}{\partial x_1 \partial x_2}\right)^2$$

be evaluated at x_0, then

(a) $\Delta > 0$ and $\partial^2 f / \partial x_1 \partial x_1 > 0$ imply f has a local minimum at x_0.
(b) $\Delta > 0$ and $\partial^2 f / \partial x_1 \partial x_1 < 0$ imply f has a local maximum at x_0.
(c) $\Delta < 0$ implies x_0 is a saddle point of f.

3. Question: Taylor Polynomial and Taylor-Series *(elementary)*

3.1. Show that the Taylor Series generated by the function $f(x) := e^x$ at $x_0 = 0$ converges to $f(x)$ for every value of x.

3.2. Use your result form **3.1** to prove that $e = \sum_{n=0}^{\infty} \frac{1}{n!}$.

In fact, it turns out that e^x *is analytic in the sense that the Taylor series* $\sum_{k=0}^{\infty} \frac{e^{x_0}}{k!} (x - x_0)^k$ *converges to* e^x *for all* $x_0 \in \mathbb{R}$*. However, even a converging Taylor series does not necessarily converge to its corresponding function* $f(x)$*.*

3.3. Find the Taylor series approximation of

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

at $x_0 = 0$. How accurate is the k th degree Taylor approximation?

Hint: You may use the fact that $\lim_{x \to \infty} \frac{e^{-\frac{1}{x^2}}}{x^n} = 0 \; \forall n \in \mathbb{N}$.

If you have any questions or feedback, please feel free to contact me via E-mail at hannah.kuempel@stat.uni-muenchen.de!!

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