1. Question: Convergence of Sequences *(elementary)*

1.1. Show that $\lim_{n\to\infty} \frac{n^2}{n^2+n+1} = 1$ using proposition 6.1.

1.2. Prove the following statement: *If $\{x_n\}$ is convergent, then $\{x_n\}$ is bounded.*
Hint: Here, it might help you to set $\varepsilon = 1$ and separately consider the cases $n \leq M$ and $n > M$ for some $M \in \mathbb{N}_{>0}$.

1.3. In each of the following cases, decide whether the sequence is convergent or divergent. If convergent, find its limit. You may use the following fact:

If $c \in (0,1)$, then $\underset{n\to\infty}{c^n} = 0$. If $c > 1$, then $\{c_n\}$ is unbounded and diverges.

(i) $a_n = 5 - 0.1^n$

(ii) $a_n = 1^n + (-1)^n$

(iii) $a_n = \frac{\sin n}{n}$

(iv) $a_n = \frac{2-n}{7+3n}$

(v) $a_n = \frac{3^{n+1}}{2^{2n+1}}$

(vi) $a_n = \frac{3^{n-1}}{2^{n+3}}$

2. Question: Convergence of series *(elementary)*

2.1. Prove the following statement: *If $|r| \geq 1$, then $\sum_{n=0}^{\infty} r^n$ diverges.*

2.2. *Using the result from **2.1** and the identity*

$$\sum_{n=0}^{m} r^n = \frac{1-r^{m+1}}{1-r},$$

*prove that the so-called **geometric series** $\sum_{n=0}^{\infty} \alpha(r)^n$, with some scalar α, converges to $\frac{\alpha}{1-r}$ if and only if $|r| < 1$.*

2.3. In each of the following cases, decide whether the series is convergent or divergent. If convergent, find its limit.

(i) $\sum_{n=0}^{\infty} \frac{2}{3^n}$

(ii) $\sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^{n+1}$

(iii) $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

(iv) $\sum_{n=1}^{\infty} \frac{1}{n}$ (*Attention: The answer might not be what you think*)

3. Question: Limits inferior and superior (*elementary*)

3.1. Calculate the limit superior and limit inferior for the following sequences.

(a) $x_n = \frac{1}{n}$

(b) $x_n = (-1)^n$

3.2. Do the sequences (a) and (b) from above question converge? If so, name their limit.

3.3. The following statement is an important fact:

Let $\{x_n\}$ be a bounded sequence. Then, $\{x_n\}$ converges if and only if $\liminf x_n = \limsup x_n$.

Direction ($\Longrightarrow$) may be proven via subsequences (see the lecture named under *Helpful Additional Resources*). Prove the other direction, i.e. $\liminf x_n = \limsup x_n \Longrightarrow \{x_n\}$ converges.

4. Question: Differentiability of real-valued functions (*elementary*)

4.1. Prove that for the function $f(x) = ax + b$,

$$f'(c) = a \quad \forall c \in \mathbb{R}.$$

4.2. Is the function

$$f : \mathbb{R} \to \mathbb{R}, \quad x \mapsto \begin{cases} 1/x & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

differentiable at $x \neq 0$? What about $x = 0$?

4.3. Use the Mean Value Theorem to prove the following statement:
If $f : I \to \mathbb{R}$ is differentiable and $f'(x) = 0$ for all $x \in I$, then f is constant.

5. Question: Differentiability and Continuity (*slightly advanced*)

Recall that we already defined the concept of continuous functions in the first Tutorial session using the **epsilon-delta criterion**. For a function $f : S \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ we can re-write it as follows:

The function f is continuous on S if

$\forall c \in S$ and $\forall \epsilon > 0, \exists \delta = \delta(\epsilon, c) > 0$ such that $\forall x \in S, |x - c| < \delta \Longrightarrow |f(x) - f(c)| < \epsilon$.

Here, $\delta(\epsilon, c)$ denotes the fact that δ can depend on ϵ and c.

5.1. Does a function $f : S \subseteq \mathbb{R} \to \mathbb{R}$ being continuous imply that f is differentiable? Prove your answer using the function $f(x) := |x|$.
Hint: You might need the reverse triangle inequality: $||x| - |x_0|| \leq |x - x_0|$.

5.2. Show that the converse it true, i.e. if $f : S \subseteq \mathbb{R} \to \mathbb{R}$ is differentiable at $c \in S$, then f is continuous at c.

If you have any questions or feedback, please feel free to contact me via E-mail at hannah.kuempel@stat.uni-muenchen.de!!

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