1. Question: Inner product and orthonormal basis of $\mathbb{R}^n$ (*elementary*)

1.1. Show that the dot product for vectors, i.e. $\mathbf{x}^\top \mathbf{y}$ with $\mathbf{x}, \mathbf{y} \in V$ for some vector space V, is an inner product. Which norm and metric does it induce?

1.2. Can you think of a definition of an inner product $\langle \cdot, \cdot \rangle_{\text{ex}}$ that isn't the dot product $\langle \cdot, \cdot \rangle_{\text{dot}}$? If two vectors are orthogonal in $(V, \langle \cdot, \cdot \rangle_{\text{dot}})$, are they also orthogonal in $(V, \langle \cdot, \cdot \rangle_{\text{ex}})$?

1.3. Show that, for some inner product space $(V, \langle \cdot, \cdot \rangle)$ with zero element $\mathbf{0}$, the following holds: *All vectors* $\mathbf{v} \in V$ *are orthogonal to* $\mathbf{0}$*, and* $\mathbf{0}$ *is the only vector in* V *that is orthogonal to itself.*

2. Question: Angles between Vectors and Projection onto a Line (*elementary*)

2.1. Find the angle in between vectors $\mathbf{a} = (8, -2, 16)^\top$ and $\mathbf{b} = (-9, 8, 12)^\top$ in radian and degrees.

2.2. Calculate the angle between the vectors $\boldsymbol{x} = [1, 1]^\top, \boldsymbol{y} = [-1, 1]^\top \in \mathbb{R}^2$ with regards to *both the dot product and the inner product defined as*

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle := \boldsymbol{x}^\top \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \boldsymbol{y}\,.$$

2.3. Let $\mathbf{y} = \begin{bmatrix} 7 \\ 6 \end{bmatrix}$ and $\mathbf{u} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$. Find the orthogonal projection of $\mathbf{y}$ onto u. Then write $\mathbf{y}$ as the sum of two orthogonal vectors, one in span$\{\mathbf{u}\}$ and one orthogonal to $\mathbf{u}$.

2.4. Project

(i) the vector $\begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix}$ orthogonally onto the line $\left\{ c \begin{pmatrix} -3 \\ 1 \\ -3 \end{pmatrix} \middle| \, c \in \mathbb{R} \right\}$

(ii) $\begin{pmatrix} -1 \\ -1 \end{pmatrix}$ orthogonally onto the line $y = 3x$.

3. Question: Gram-Schmidt Process (*elementary*)

3.1. Carry out the Gram-Schmidt orthonormalization process on the following pair of vectors in $\mathbb{R}^2$ to obtain an orthonormal basis:

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} -1 \\ 3 \end{bmatrix}.$$

3.2. Carry out the Gram-Schmidt orthonormalization process on the following three vectors in $\mathbb{R}^3$ to obtain an orthonormal basis:

$$\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 8 \\ 1 \\ -6 \end{bmatrix}, \text{ and } \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

3.3. Perform an Eigendecomposition of the following matrix. How do the spectral theorem and the Gram-Schmidt process help you here?

$$\boldsymbol{A} = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & 2 \\ 2 & 2 & 3 \end{bmatrix}.$$

(You may use the fact that $\det(\mathbf{A} - \lambda \mathbf{I}) = -(\lambda - 1)^2(\lambda - 7)$ to save time.)

4. Question: Projection onto general Subspaces (*elementary*)

4.1. How does the formula for orthogonal projection onto subspaces from definition 5.7 simplify if the given basis is not only orthogonal but orthonormal?

4.2. In $\mathbb{R}^3$, let

$$W = \text{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \right\}$$

be the subspace spanned by the vectors $(1,1,2)^\top$ and $(1,1,-1)^\top$. What point of W is closest to the vector $(4,5,-2)^\top$?

4.3. In $\mathbb{R}^3$, find the orthogonal projection of $(2,2,5)^\top$ on the subspace

$$W = \text{span}\left\{ \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} \right\}.$$

4.4. Let $\{\mathbf{w_1}, \mathbf{w}_2, \ldots, \mathbf{w}_m\}$ be an orthonormal basis of the subspace $W \subset V$. Prove that the vectors $(v - \text{pr}_W(v))$ and $\mathbf{w}_k$ ar orthogonal $\forall v \in V$, $k = 1, \ldots, m$; and hence $v - \text{pr}_W(v)$ is orthogonal to every vector in W.

If you have any questions or feedback, please feel free to contact me via E-mail at hannah.kuempel@stat.uni-muenchen.de!!

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