1. Question: Determinants (*elementary*)

1.1. What are the determinants of the following matrices

$$A = \begin{bmatrix} 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}?$$

1.2. Compute the determinants of

$$A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 3 & 1 & 4 & 5 & 6 \\ 0 & 3 & 2 & 7 & 1 & 8 \\ 0 & 0 & 1 & 2 & 4 & 3 \\ 0 & 0 & 0 & 1 & 5 & 2 \\ 0 & 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix} .$$

1.3. Use a cofactor expansion across the third row to compute $\det(A)$, where

$$A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$$

1.4. Compute $\det(A)$, where

$$A = \begin{bmatrix} 3 & -7 & 8 & 9 & -6 \\ 0 & 2 & -5 & 7 & 3 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 2 & 4 & -1 \\ 0 & 0 & 0 & -2 & 0 \end{bmatrix}$$

2. Question: Eigenspaces

2.1. Name all eigenvalues and corresponding eigenspaces of the identity matrix $\mathbf{I}_n$ in $\mathbb{R}^{n\times n}$, $n \in \mathbb{N}_{>0}$. (*elementary*)

2.2. Compute all eigenvalues and corresponding eigenspaces of the following matrices: (*basic*)

$$\boldsymbol{A} = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 4 & 0 & 1 \\ -1 & -6 & -2 \\ 5 & 0 & 0 \end{bmatrix}$$

2.3. Consider a vector $v \in \mathbb{R}^n$, $n \in \mathbb{N}_{>0}$. Name all eigenvectors and corresponding eigenspaces of $vv^\top$. (*basic*)

3. Question: Conceptual questions (*basic*)

For some questions in this exercise, you will need the following: *The determinant of a square matrix is equal to the product of its eigenvalues, while the trace is equal to the sum of its eigenvalues.* (For a proof, see page 3 of `https://www.adelaide.edu.au/mathslearning/ua/media/120/evalue-magic-tricks-handout.pdf`)

3.1. Prove that a square matrix A is invertible if and only if it has no eigenvalues of value 0.

3.2. If an invertible matrix A has eigenvalues $\lambda_1, \ldots, \lambda_n$ then what are the eigenvalues of A^{-1}? What about $A + I$? What about $A^\top$? What about A^2? What about A^3? Answer the same question for eigenvectors.

3.3. Is it possible to determine the eigenvalues of the following matrices?

(i) A, a 2×2 matrix with $\det(A) = -2$ and $\operatorname{tr}(A) = 0$.

(ii) B, a 3×3 matrix with $\det(A) = -2$ and $\operatorname{tr}(A) = 0$.

3.4. Prove that if matrices A and B are similar then they have identical eigenvalues.

3.5. Prove the following statement: *Similar matrices are always equivalent. However, equivalent matrices are not necessarily similar.*

4. Question: Matrix decomposition (*basic*)

In this question, you may use the following: *A square matrix is called **orthogonal**, if its columns are*

- *orthonormal to each other (i.e. the product of any two columns is* 0*)*
- *of unit length (have length* 1*).*

For an orthogonal matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$*, it holds that* $\mathbf{A}^{-1} = \mathbf{A}^\top$.

4.1. Compute the eigendecomposition of $\boldsymbol{A} = \frac{1}{2}\begin{bmatrix} 5 & -2 \\ -2 & 5 \end{bmatrix}$.

4.2. Another popular matric decomposition is the Cholesky decomposition. It says that any symmetric matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$ with only strictly positive eigenvalues may be decomposed as follows:

$$\boldsymbol{A} = \boldsymbol{L}\boldsymbol{L}^\top = \begin{pmatrix} l_{11} & \dots & 0 \\ \vdots & \ddots & \vdots \\ l_{n1} & \dots & l_{nn} \end{pmatrix} \begin{pmatrix} l_{11} & \dots & l_{1n} \\ \vdots & \ddots & \vdots \\ 0 & \dots & l_{nn} \end{pmatrix}.$$

For the following decomposition of a 3×3 matrix, write all l_{ij}s, in terms of the a_{ij}s, $i, j = 1, 2, 3$.

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{21} & a_{22} & a_{32} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \boldsymbol{L}\boldsymbol{L}^\top = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{bmatrix}.$$

4.3. Write out the matrix decompositions for matrices A and B from **2.2** (and check whether the decomposition truly results in A and B if you want to practice matrix multiplication).

If you have any questions or feedback, please feel free to contact me via E-mail at hannah.kuempel@stat.uni-muenchen.de!!

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