1. Question: Basic questions (*elementary*)

1.1. Think of **two different bases** of $\mathbb{R}^3$ that are <u>not</u> of the form $\left\{ \begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ b \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ c \end{pmatrix} \right\}$ for any $a, b, c \in \mathbb{R}$.

1.2. Determine the rank of the following 3 matrices:

$$A = \begin{pmatrix} 1 & 2 & 1 \\ -2 & -3 & 1 \\ 3 & 5 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \text{ and } \quad C = \begin{pmatrix} 1 & 2 & 5 \\ 0 & 4 & 8 \\ 2 & 3 & 8 \\ 5 & 2 & 9 \end{pmatrix}.$$

1.3. Find the basis of the vector subspace $U \subseteq \mathbb{R}^5$, spanned by the vectors

$$\boldsymbol{x}_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \\ -1 \\ -1 \end{bmatrix}, \quad \boldsymbol{x}_2 = \begin{bmatrix} 2 \\ -1 \\ 1 \\ 2 \\ -2 \end{bmatrix}, \quad \boldsymbol{x}_3 = \begin{bmatrix} 3 \\ -4 \\ 3 \\ 5 \\ -3 \end{bmatrix}, \quad \boldsymbol{x}_4 = \begin{bmatrix} -1 \\ 8 \\ -5 \\ -6 \\ 1 \end{bmatrix} \in \mathbb{R}^5.$$

1.4. Consider $V = \{(a_1, a_2) : a_1, a_2 \in \mathbb{R}\}$. Is V a vector space for the following definitions, where $a_1, a_2, b_1, b_2, c \in \mathbb{R}$, of addition and scalar multiplication? Justify your answer.

(i) $(a_1, a_2) + (b_1, b_2) := (a_1 + 2b_1, a_2 + 3b_2)$ and $c\,(a_1, a_2) := (ca_1, ca_2)$.

(ii) $(a_1, a_2) + (b_1, b_2) := (a_1 + b_1, a_2 + b_2)$ and $c\,(a_1, a_2) = \begin{cases} (0,0) & \text{if } c = 0 \\ (ca_1, \frac{a_2}{c}) & \text{if } c \neq 0 \end{cases}$.

2. Question: Linear system of equations (*basic*)

Let A be the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 2 \\ 2 & 2 & 3 & 3 \end{bmatrix}$$

2.1. Determine if the system $A\mathbf{x} = \mathbf{0}$ has zero, one or infinitely many solutions, and compute a basis of the space of solutions.

2.2. Is it true that the system $A\mathbf{x} = \mathbf{b}$ has a solution for any $b \in \mathbb{R}^3$? If so, prove the statement, otherwise find a counterexample.
Hint: For $A\mathbf{x} = \mathbf{b}$ to have a solution, both A and the augmented matrix $[A|\mathbf{b}]$ need to be of the same rank.

2.3. Prove the following statement: *The null space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^n$. Equivalently, the set of all solutions to a system $A\mathbf{x} = \mathbf{0}$ of m homogeneous linear equations in n unknowns is a subspace of $\mathbb{R}^n$.* (***less basic***)

3. Question: Intersection and Addition of Subspaces (*slightly more advanced*)

Let $V \subseteq \mathbb{R}^4$ be the subspace $V = \text{span}\,(v_1, v_2)$, where

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad v_2 = \begin{bmatrix} 1 \\ 3 \\ 1 \\ 0 \end{bmatrix}$$

and let $W \subseteq \mathbb{R}^4$ be the subspace given by the solutions of the system

$$\begin{cases} x_1 + x_2 + 2x_4 = 0 \\ 2x_1 + x_2 - x_3 = 0. \end{cases}$$

Find a basis of $V \cap W := \{u : u \in V \text{ AND } u \in W\}$ and a basis of $V + W := \{u = v + w : \ v \in V, w \in W\}$.
Hint: A vector $x \in \mathbb{R}^4$ belongs to V if and only if the two matrices $[v_1 \ v_2]$ and $\big[[v_1 \ v_2] \,\big|\, x \big]$ have the same rank.

4. Question: Linear Maps (*slightly more advanced*)

4.1. Let $f : \mathbb{R}^2 \to \mathbb{R}^3$ be the linear map defined as

$$f\left(\begin{bmatrix} x \\ y \end{bmatrix} \right) = \begin{bmatrix} x + 2y \\ 2x + 4y \\ x + ay \end{bmatrix}$$

where $a \in \mathbb{R}$ is a parameter. Find the matrix $[f]$ associated to f with respect to the standard bases of $\mathbb{R}^2$ and $\mathbb{R}^3$.

4.2. Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear map such that

$$f\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}, \quad f\left(\begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 6 \\ 8 \\ 10 \end{bmatrix} \quad \text{and} \quad f\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 10 \\ 14 \\ 18 \end{bmatrix}.$$

(i) Compute the dimensions of $\ker f$ and $\operatorname{Im} f$.

(ii) For f from **4.2.**, compute $f\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right)$.

5. Question: Vector space of real-valued functions (*Tedious, but definitely a fact to remember even if you don't want to verify*)

Prove that the set of all real-valued functions $f : \mathbb{R} \to \mathbb{R}$ is a vector space, if addition and scalar multiplication is defined as follows, for $c \in \mathbb{R}$:

$$(f + g)(x) = f(x) + g(x), \quad (cf)(x) = cf(x)\,.$$

If you have any questions or feedback, please feel free to contact me via E-mail at hannah.kuempel@stat.uni-muenchen.de!!

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