1. Question: Intuition for those unfamiliar (*very elementary*)

1.1. Write out the transpose of the following matrix:

$$\mathbf{A} = \begin{pmatrix} 1 & 3 & 5 & 7 & 1 \\ 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 10 \end{pmatrix}$$

1.2. Write out two matrices that cannot be multiplied with each other.

1.3. Write 2×2 matrices A and B such that $AB \neq BA$. Verify your solution by computing the products.

1.4. Write 2×2 matrices A, B, C such that $AB = AC$ but $B \neq C$. Verify your solution by computing the products.

1.5. Find two matrices that are inverse to each other.

1.6. Find (and sketch) a system of linear equations each, that

– has infinitely many solutions
– has exactly one solution
– has no solution.

(Of course, you shouldn't use the examples from the booklet.)

2. Question: Vector and Matrix Multiplication (*elementary, but good computational practice!*)

2.1. Given the vectors $\mathbf{a}$ and $\mathbf{b}$:

$$\mathbf{a} = \begin{pmatrix} 1 \\ 0 \\ 5 \\ 7 \end{pmatrix} \quad \& \quad \mathbf{b} = \begin{pmatrix} 2 \\ 4 \\ 0 \\ 8 \end{pmatrix}$$

Compute the following:

(i) the four products $\mathbf{a}^T\mathbf{b}$; $\mathbf{b}^T\mathbf{a}$; $\mathbf{a}^T\mathbf{b}\mathbf{a}$; $\mathbf{b}^T\mathbf{a}\mathbf{b}$
(*Hint: This should only be three different values*)

(ii) the two products $\mathbf{a}\mathbf{b}^T$ and $\mathbf{b}\mathbf{a}^T$. Do you notice anything here?

2.2. Given the following matrix A and vector $\mathbf{b}$:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \quad \& \quad \mathbf{b} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

compute the two products $A\mathbf{b}$ and $\mathbf{b}^\top A$.

2.3. Given the matrices

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 5 & 0 \end{pmatrix} \quad \& \quad \mathbf{B} = \begin{pmatrix} 0 & 2 \\ 4 & 0 \\ 0 & 6 \end{pmatrix}$$

compute the product $\mathbf{AB}$.

3. Question: Verification of Matrix Properties (*basic*)

3.1. Verify the **Associativity** and **Distributivity** properties of real-valued matrix operations (i.e. show that they are true). (***elementary***)

3.2. Show that Associativity and Distributivity also hold for multiplying a scalar with real-valued matrixes, i.e. that the following statements are true $\forall \lambda, \psi \in \mathbb{R}$:

(i) Associativity:

$$(\lambda\psi)\boldsymbol{C} = \lambda(\psi\boldsymbol{C}), \quad \boldsymbol{C} \in \mathbb{R}^{m\times n}$$

$$\text{and } \lambda(\boldsymbol{BC}) = (\lambda\boldsymbol{B})\boldsymbol{C} = \boldsymbol{B}(\lambda\boldsymbol{C}) = (\boldsymbol{BC})\lambda, \quad \boldsymbol{B} \in \mathbb{R}^{m\times n}, \boldsymbol{C} \in \mathbb{R}^{n\times k}.$$

(ii) Distributivity:

$$\begin{aligned} (\lambda+\psi)\boldsymbol{C} &= \lambda\boldsymbol{C} + \psi\boldsymbol{C}, & \boldsymbol{C} \in \mathbb{R}^{m\times n} \\ \text{and } \lambda(\boldsymbol{B}+\boldsymbol{C}) &= \lambda\boldsymbol{B} + \lambda\boldsymbol{C}, & \boldsymbol{B},\boldsymbol{C} \in \mathbb{R}^{m\times n} \end{aligned}$$

(***elementary***)

3.3. Prove that if a square matrix $\boldsymbol{A}$ is invertible, its inverse $\boldsymbol{A}^{-1}$ is unique.
Hint: *Generally, one can prove uniqueness by considering two elements that satisfy the given property (here, being the inverse of* $\boldsymbol{A}$*) and show that the two elements are equal to each other.*
(***slightly less elementary***)

3.4. Prove that the following holds for two non-zero $n \times n$, $n \in \mathbb{N}_{>0}$ matrices A, B and **invertible** $n \times n$ matrix C

$$AC = BC \Longrightarrow A = B \quad \text{and} \quad CA = CB \Longrightarrow A = B\,.$$

(***less elementary***)

4. Question: Systems of Linear Equations (*basic*)

4.1. Find all solutions of the following system of linear equations. (***elementary***)

$$\begin{aligned} 4x_2 + 8x_3 &= 12 \\ x_1 - x_2 + 3x_3 &= -1 \\ 3x_1 - 2x_2 + 5x_3 &= 6 \end{aligned}$$

4.2. Consider the following system of linear equations:

$$\begin{array}{rcr} -2x_1 + 4x_2 - 2x_3 - x_4 + 4x_5 &=& -3 \\ 4x_1 - 8x_2 + 3x_3 - 3x_4 + x_5 &=& 2 \\ x_1 - 2x_2 + x_3 - x_4 + x_5 &=& 0 \\ x_1 - 2x_2 - 3x_4 + 4x_5 &=& a \end{array}.$$

For which $a \in \mathbb{R}$ can it be solved? Give **one** particular solution to this linear system. (***slightly less elementary***)

4.3. Let $\mathbf{s}_p \in \mathbb{R}^n$ be a (particular) solution to a system of linear equations defined by $\mathbf{Ax} = \mathbf{b}$, with $\mathbf{A} \in \mathbb{R}^{n\times n}$, $\mathbf{b} \in \mathbb{R}^n$. Furthermore, consider the set

$$S = \{\mathbf{s} \in \mathbb{R}^n \mid \mathbf{s} \text{ is a solution to the linear system } \mathbf{Ax} = \mathbf{0}_n\}\,.$$

Prove that $\forall \mathbf{s} \in S$, $\mathbf{s}_p + \mathbf{s}$ is a solution to $\mathbf{Ax} = \mathbf{b}$.
What are the possible sizes of the sets of all possible solutions for any given linear system? (***slightly more challenging***)

5. Question: Inverse of a matrix (*slightly more challenging*)

5.1. To compute the inverse $\boldsymbol{A}^{-1}$ of $\boldsymbol{A} \in \mathbb{R}^{n \times n}$, we need to find a matrix $\boldsymbol{A}^{-1}$ that satisfies $\boldsymbol{A}\boldsymbol{A}^{-1} = \boldsymbol{I}_n$. We can do this by transforming $[\boldsymbol{A} \mid \boldsymbol{I}_n]$ to $\left[\boldsymbol{I}_n \mid \boldsymbol{A}^{-1}\right]$, specifically by applying Gaussian elimination until the left side is the identity matrix, in which case the right side give the inverse.

Apply this principle to calculate the inverse of the following matrix:

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

5.2. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a non-zero 2×2 matrix. Show that

(1.) If $ad - bc = 0$, then A has no inverse and

(2.) If $ad - bc \neq 0$, then

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

6. Question: *Freaky Fun*

In math, computing, etc., the modulo operation gives us the remainder when one integer is divided by another integer. Specifically, for $a, n \in \mathbb{Z}$, we have

$$a \mod n := a - n \left\lfloor \frac{a}{n} \right\rfloor.$$

Now, consider the set $\mathbb{F}_7 = \{1, 2, 3, 4, 5, 6\}$ with the following two operations:

$$+ : \mathbb{F}_7 \times \mathbb{F}_7 \to \mathbb{F}_7, (x, y) \longmapsto x + y \bmod 7$$
$$\cdot : \mathbb{F}_7 \times \mathbb{F}_7 \to \mathbb{F}_7, (x, y) \longmapsto xy \bmod 7$$

(*In case this reminds you of quotient spaces you're not wrong, but we won't even talk about vector spaces until next week.*)

Assuming that addition and multiplication of matrices with entries in $\mathbb{F}_7$ is defined analogously to the real-valued case, calculate $A + B$; AB ; and BA for matrices

$$A = \begin{pmatrix} 0 & 2 & 4 \\ 1 & 5 & 6 \\ 0 & 1 & 0 \end{pmatrix} \quad \& \quad B = \begin{pmatrix} 1 & 1 & 0 \\ 5 & 3 & 1 \\ 0 & 4 & 6 \end{pmatrix}.$$

If you have any questions or feedback, please feel free to contact me via E-mail at hannah.kuempel@stat.uni-muenchen.de!!

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