1. Question: Conditional expectation (*elementary*)

1.1. Suppose we draw $X \sim \text{Unif}(0,1)$. After we observe $X = x$, we draw $Y \mid X = x \sim \text{Unif}(x,1)$, resulting in the conditional density $f_{Y|X}(y \mid x) = 1/(1-x)$ for $x < y < 1$.
Find the conditional expectation of Y given X.

1.2. Let $X \sim$ Uniform $(0,1)$. Let $0 < a < b < 1$. Consider

$$Y = \begin{cases} 1 & 0 < x < b \\ 0 & \text{otherwise,} \end{cases} \qquad \text{and} \qquad Z = \begin{cases} 1 & a < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

(i) Are Y and Z independent? Why/Why not?

(ii) Find $\mathbb{E}(Y|Z)$. Hint: What values z can Z take? Now find $\mathbb{E}(Y \mid Z = z)$.

1.3. Let $r(x)$ be a function of x and let $s(y)$ be a function of y. Show that

$$\mathbb{E}[r(X)s(Y) \mid X] = r(X)\mathbb{E}[s(Y) \mid X].$$

2. Question: Conditional variance and law of total variance (*medium*)

2.1. The Bernoulli distribution with parameter p is defined via the pmf $f(x) = p^x(1-p)^{1-x}, \quad x \in \{0,1\}$.
Consider two random variables $X, Y \sim \text{Bernoulli}\left(\frac{2}{5}\right)$ with

$$X \mid Y = 0 \sim \text{ Bernoulli}\left(\frac{2}{3}\right), \quad P(X = 0 \mid Y = 1) = 1, \quad \text{Var}(\mathbb{E}[X|Y]) = \frac{8}{75}.$$

Find the pmf of $V := \text{Var}(X|Y)$, $\mathbb{E}[V]$, and check that $\text{Var}(X) = \mathbb{E}[V] + \text{Var}(\mathbb{E}[X|Y])$.

2.2. Consider a random variable N that takes values in $\mathbb{N}$ and suppose that we know $\mathbb{E}[N]$ and $\text{Var}(N)$. Find the expectation and variance of the random variable

$$Y = \sum_{i=1}^{N} X_i,$$

where the X_i are i.i.d. and also independent of N.
Hint: You may use the fact that for independent X, Y, Z, we have $\mathbb{E}[X + Y|Z] = \mathbb{E}[X|Z] + \mathbb{E}[Y|Z] = \mathbb{E}[X] + \mathbb{E}[Y]$, with both equality following immediately from independence.

3. Question: Properties of the conditional expectation (*slightly advanced*)

Let $X, Y : (\Omega, \mathcal{F}) \longrightarrow (\Omega', \mathcal{F}')$ be random variables with $\mathbb{E}[X], \mathbb{E}[Y] < \infty$.

3.1. Show that if a and b are constants and $\mathcal{A} \subset \mathcal{F}$, then $E(aX + bY \mid \mathcal{A}) = aE(X \mid \mathcal{A}) + bE(X \mid \mathcal{A})$ a.s.

3.2. Show that if $X \leq Y$ a.s., then, for $\mathcal{A} \subset \mathcal{F}$, $E(X \mid \mathcal{A}) \leq E(Y \mid \mathcal{A})$ a.s.
Hint: This can be accomplished by showing that $P(\{E(X \mid \mathcal{A}) > E(Y \mid \mathcal{A})\}) = 0$.

3.3. Let $\mathcal{A}$ and $\mathcal{A}_0$ be σ-algebras satisfying $\mathcal{A}_0 \subset \mathcal{A} \subset \mathcal{F}$. Show that

$$E\left[E(X \mid \mathcal{A}) \mid \mathcal{A}_0\right] = E\left(X \mid \mathcal{A}_0\right) = E\left[E\left(X \mid \mathcal{A}_0\right) \mid \mathcal{A}\right] \text{ a.s.}$$

If you have any questions or feedback, please feel free to contact me via E-mail at hannah.kuempel@stat.uni-muenchen.de!!

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