1. Question: Basic Inequalities (*very elementary*)

Let X be an exponential random variable with parameter $\lambda = 12$, i.e. with density

$$f_X(x) = \begin{cases} 12e^{-12x}, & x \geq 0, \\ 0, & x < 0. \end{cases}$$

1.1. Use Markov's inequality to find an upper bound for $P(X > 6)$.

1.2. Use Chebyshev's inequality to find an upper bound for $P(X > 6)$.

1.3. Explicitly compute the probability above and compare with the upper bounds you derived.

2. Question: Transformations of Several Random Variables (*elementary*)

2.1. Let X and Y be independent random variables with cumulative distribution functions F_X and F_Y, respectively. Show that the cumulative distribution function of $X + Y$ is

$$F_{X+Y}(t) = \int F_Y(t - x) d\mathbb{P}_X(x). \qquad (\star)$$

2.2. The concept of **2.1** is also referred to as *convolution*. Specifically write out the pmf and pdf of $X + Y$ when X and Y are discrete and continuous RVs, respectively.
Hint: You may use that in ($\star$), integration and differentiation are interchangeable by the dominated convergence theorem and mean value theorem.

2.3. Let X be a uniform distribution on $[0, 1]$, i.e. $f_X(x) = \frac{1}{1-0}\mathbb{1}_{x\in[0,1]}$, and Y be a uniform distribution on $[1, 2]$, i.e. $f_Y(x) = \frac{1}{2-1}\mathbb{1}_{x\in[1,2]}$. Find f_Z for $Z := X + Y$.

2.4. Determine the cdf of the random variable $Z := \min\{X, Y\}$ for independent random variables X and Y. What does the pdf look like if X and Y are continuous?

3. Question: Convergence of Random Variables (*elementary*)

3.1. Consider a sequence of random variables $(X_n : n \in \mathbb{N})$ such that $X_n \xrightarrow{L^p} X$, then $X_n \xrightarrow{P} X$.

3.2. Let $X_1, \ldots, X_n$ be IID with finite mean $\mu = \mathbb{E}(X_1)$ and finite variance $\sigma^2 = \mathbb{V}(X_1)$. Let $\bar{X}_n$ be the sample mean and let S_n^2 be the sample variance.

(i) Show that $\mathbb{E}[\bar{X}] = \mu$ and $\mathbb{E}\left(S_n^2\right) = \sigma^2$. (*You may use that* $E\left(\bar{X}^2\right) = \frac{\sigma^2}{n} + \mu^2$.)

(ii) Show that $S_n^2 \xrightarrow{\text{P}} \sigma^2$. *Hint: Show that* $S_n^2 = c_n n^{-1} \sum_{i=1}^n X_i^2 - d_n \bar{X}_n^2$ *where* $c_n \to 1$ *and* $d_n \to 1$. *Apply the law of large numbers to* $n^{-1} \sum_{i=1}^n X_i^2$ *and to* $\bar{X}_n$. *Then use part (e) of Theorem 11.1.*

3.3. Let $X_1, X_2, \ldots$ be a sequence of random variables such that

$$\mathbb{P}\left(X_n = \frac{1}{n}\right) = 1 - \frac{1}{n^2} \quad \text{and} \quad \mathbb{P}\left(X_n = n\right) = \frac{1}{n^2}$$

Does X_n converge in probability? Does X_n converge in L^2?

3.4. Construct an example where $X_n \rightsquigarrow X$ and $Y_n \rightsquigarrow Y$ but $X_n + Y_n$ does not converge in distribution to $X + Y$.

4. Question: Miscellaneous Probability Theory
(*slighlty more advanced*)

4.1. Let X be a random variable with $\mathbb{E}[X]^2 < \infty$ and let $Y = |X|$. Suppose that X has a Lebesgue density symmetric about 0 . Show that X and Y are uncorrelated (i.e. $\text{Cov}(X, Y) = \mathbb{E}[XY] - \mathbb{E}[X]EY = 0$), but they are not independent.

4.2. Show that a random variable X is independent of itself if and only if X is constant a.s. Can X and $f(X)$ be independent, where f is a Borel function?

4.3. Let X be a random variable having a cumulative distribution function F with corresponding probability measure $\mathbb{P}$. Show that if $\mathbb{E}[X]$ exists, then

$$\mathbb{E}[X] = \int_0^\infty [1 - F(x)]dx - \int_{-\infty}^0 F(x)dx.$$

If you have any questions or feedback, please feel free to contact me via E-mail at hannah.kuempel@stat.uni-muenchen.de!!

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