1. Question: Integration w.r.t. different measures (*elementary*)

1.1. Consider the function

$$f : \mathbb{R} \longrightarrow \mathbb{R}, \quad x \mapsto \begin{cases} 1, & \text{if } -1 < x \leq 0 \\ 2, & \text{if } 0 < x \leq 1 \\ 3, & \text{if } 1 < x \leq 2, \\ 0, & \text{otherwise.} \end{cases}$$

Calculate the integral of this function w.r.t.

(i) the Lebesgue measure and

(ii) the Dirac measure, defined on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ as $\delta_y(x) = \mathbb{1}_{x=y}$ for a fixed $y \in \mathbb{R}$.

1.2. Let μ be the counting measure on $\mathbb{N}$, and define the sequence $\{f_n\}_{n \in \mathbb{N}}$ by

$$f_n(x) = \begin{cases} 1 & \text{if } x = n \\ 0 & \text{otherwise.} \end{cases}$$

Compute

(i) $\lim_{n\to\infty} \int f_n d\mu$ and

(ii) $\int \lim_{n\to\infty} f_n d\mu$.

2. Question: Measures and Probability Space (*elementary*)

2.1. Take the measurable space $\Omega = \{1, 2\}, F = 2^{\Omega}$. Which of the following is a measure? Which is a probability measure?

a. $\mu(\emptyset) = 0, \mu(\{1\}) = 5, \mu(\{2\}) = 6, \mu(\{1, 2\}) = 11$

b. $\mu(\emptyset) = 0, \mu(\{1\}) = 0, \mu(\{2\}) = 0, \mu(\{1, 2\}) = 1$

c. $\mu(\emptyset) = 0, \mu(\{1\}) = 0, \mu(\{2\}) = 0, \mu(\{1, 2\}) = 0$

d. $\mu(\emptyset) = 0, \mu(\{1\}) = 0, \mu(\{2\}) = 1, \mu(\{1, 2\}) = 1$

e. $\mu(\emptyset) = 0, \mu(\{1\}) = 0, \mu(\{2\}) = \infty, \mu(\{1, 2\}) = \infty$

2.2. Define a probability space that could be used to model the outcome of throwing two fair 6-sided dice.

2.3. Let (X, μ) be a measure space and E a measurable subset of X. Show that for every $A \subset X$ the following holds:

$$\mu(E \cap A) + \mu(E \cup A) = \mu(E) + \mu(A).$$

2.4. Let A and B be events with probabilities $P(A) = \frac{2}{3}$ and $P(B) = \frac{1}{2}$

(i) Show that $\frac{1}{6} \leq P(A \cap B) \leq \frac{1}{2}$, and give examples to show that both extremes are possible.

(ii) Find corresponding bounds for $P(A \cup B)$.

3. Question: Sigma Algebras (*medium*)

3.1. Let A be a fixed subset of a set X. Determine the σ-algebra of subsets of X generated by $\{A\}$.

3.2. Let Ω be a non-empty set. Suppose that $\mathcal{F}_1$ and $\mathcal{F}_2$ are σ-algebras on Ω. Prove that $\mathcal{F}_1 \cap \mathcal{F}_2$ is also a σ-algebra on Ω.

3.3. Suppose that $\mathcal{F}_1$ and $\mathcal{F}_2$ are σ-algebras on Ω. Show by example that $\mathcal{F}_1 \cup \mathcal{F}_2$ may fail to be a σ-algebra.
Hint: You can consider two σ-algebras $\mathcal{F}_1$ and $\mathcal{F}_2$ on $\Omega := \{1, 2, 3\}$.

3.4. Let X be an uncountable set. (An uncountable set X is one that is not countable, i.e. there is no bijection between X and $\mathbb{N}$, meaning X has more elements than the natural numbers.)
Consider

$$\mathcal{S} = \{E \subset X : E \text{ or } E^c \text{ is at most countable } \}$$

and show that $\mathcal{S}$ is a σ-algebra and that $\mathcal{S}$ is generated by the one-point subsets of X.
Hint: It will help to apply the following identity: $\cup_{k=1}^{\infty} A_k^c \subset (\cap_{k=1}^{\infty} A_k)^c$.

4. Question: More Measure Theory (*medium*)

4.1. Show that the Lebesgue measure of rational numbers on $[0, 1]$ is 0.

4.2. Take the measure space $(\Omega_1 = (0, 1], \mathcal{B}((0, 1]), \lambda)$ (we know that this is a probability space on $(0, 1]$).

(i) Define a map (function) from Ω_1 to $\Omega_2 = \{1, 2, 3, 4, 5, 6\}$ such that the measure space $\left(\Omega_2, 2^{\Omega_2}, \lambda \circ f^{-1}\right)$ will be a discrete probability space with uniform probabilities $\left(P(\omega) = \frac{1}{6}, \forall \omega \in \Omega_2\right)$.

(ii) Is the map that you defined in (i) the only such map?

(iii) How would you in the same fashion define a map that would result in a probability space that can be interpreted as a coin toss with probability p of heads?

4.3. Let $\Omega_1 = (0, 1)$, let $\mathcal{F}_1$ be the Borel sets, and let $\mathbb{P}_1$ be the Lebesgue measure. Let $\Omega_2 = (0, 1)$ let $\mathcal{F}_2$ be the set of all subsets of $(0, 1)$, and let $\mathbb{P}_2$ be the counting measure. In particular, for every infinite (countable or uncountable) subset of $(0, 1), \mathbb{P}_2(A) = \infty$.

Define

$$f : \mathbb{R}^2 \longrightarrow \mathbb{R}, \quad (x, y) = \begin{cases} 1, & \text{if } x = y \\ 0, & \text{otherwise.} \end{cases}$$

Does Fubini's theorem apply here?

5. Question: Infinite Monkey Theorem (*Freaky Fun*)

Prove the following statement: *Consider an infinite string of letters $a_1 a_2 \cdots a_n \cdots$ produced from a finite alphabet (of, say, 26 letters) by picking each letter independently at random, and uniformly from the alphabet (so each letter gets picked with probability $\frac{1}{26}$). Fix a string S of length m from the same alphabet (which is the given "text"). Let E_j be the event that the substring $a_j a_{j+1} \cdots a_{j+m-1}$ is S. Then with probability 1, infinitely many of the E_j's occur.*

If you have any questions or feedback, please feel free to contact me via E-mail at hannah.kuempel@stat.uni-muenchen.de!!

Also, thank you to Prof. Francesca Da Lio, Prof. Dr. Beatrice Acciaio, and the authors of this MATH 3160 - Probability lecture as well as the book *Principles of Uncertainty – Exercises*, Retrieved January 6, 2025, from `https://fri-datascience.github.io/course_pou/` whose exercises this sheet was inspired by.