# 1. Question: Metrics on $\mathbb{R}$

**1.1.** What would be the natural norm and metric to define on the real line? Use this metric to show that the real line is a metric space.

**1.2.** Are the following functions metrics on $\mathbb{R}$? Prove your answer.

*(i)* $d : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}, \quad (x, y) \mapsto (x - y)^2$
*(ii)* $d : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}, \quad (x, y) \mapsto \sqrt{|x - y|}$.

**1.3.** Let $d$ be a metric on $X$. Determine all constants $k \in \mathbb{R}$ such that each of the following functions $d' : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ is a metric on $X$.

*(i)* $d'(x, y) := kd(x, y)$
*(ii)* $d'(x, y) := d(x, y) + k$.

# 2. Question: Metrics on $\mathbb{R}^m$

**2.1.** Sketch the unit-ball, i.e. the $\varepsilon$-Ball in $\mathbb{R}^2$ with $\varepsilon = 1$, for the following three metrics:

- Euclidean distance, i.e.

$$d_{\text{Euclidean}} : \mathbb{R}^m \times \mathbb{R}^m \longrightarrow \mathbb{R}_{\geq 0}, \quad (\boldsymbol{x}, \boldsymbol{y}) \mapsto \sqrt{\sum_{i=1}^{m} (x_i - y_i)^2}$$

- Manhattan distance, i.e.

$$d_{\text{Manhattan}} : \mathbb{R}^m \times \mathbb{R}^m \longrightarrow \mathbb{R}_{\geq 0}, \quad (\boldsymbol{x}, \boldsymbol{y}) \mapsto \sum_{i=1}^{m} |x_i - y_i|$$

- Chebyshev distance, i.e.

$$d_{\text{Chebyshev}} : \mathbb{R}^m \times \mathbb{R}^m \longrightarrow \mathbb{R}_{\geq 0}, \quad (\boldsymbol{x}, \boldsymbol{y}) \mapsto \max\left\{|x_1 - y_1|, |x_2 - y_2|, \ldots, |x_m - y_m|\right\}.$$

**2.2.** Use the sketch from **2.1** to explain the intuition of each metric and give an example of when it might be useful.

**2.3.** is the squared Euclidean distance $d^2_{\text{Euclidean}}$ a metric? Prove your answer.

# 3. Question: Triangle inequality

**3.1.** Prove the generalized triangle inequality, i.e. that for some metric space $(X, d)$, $n > 2$, and $x_1, \ldots, x_{n-1}, x_n \in X$, it holds that

$$d(x_1, x_n) \leq d(x_1, x_2) + d(x_2, x_3) + \ldots + d(x_{n-1}, x_n) .$$

**3.2.** Using the triangle inequality, show that for any metric $d$

$$|d(x, z) - d(y, z)| \leq d(x, y) .$$

**3.3.** Using the triangle inequality, show that for any metric $d$

$$|d(x, y) - d(z, w)| \leq d(x, z) + d(y, w) .$$

# 4. Question: Open sets

**4.1.** Are the following sets open or closed in the metric spaces $(\mathbb{R}^2, d_{\text{Euclidean}})$ and $(\mathbb{R}, d_{\text{Euclidean}})$, respectively? Prove your answer.

(a) $A = \left\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 2y\right\}$

(b) $B = \left\{x \in \mathbb{R} : x^3 + 2x^2 - 3x \leq 0\right\}$

**4.2.** Suppose $(X, d)$ is a metric space and $f : X \to \mathbb{R}$ is continuous. Show that $B = \{x \in X : |f(x)| < r\}$ is open in $X$ for each $r > 0$.

**4.3.** Show that every function $f : X \to Y$ is continuous when $X, Y$ are metric spaces and the metric on $X$ is the so-called discrete metric, defined as

$$d : X \times X \longrightarrow \{0, 1\},\ (x, y) \mapsto \mathbb{1}_{x \neq y} := \begin{cases} 1, & x \neq y, \\ 0, & \text{otherwise, i.e. } x = y \end{cases}.$$

**4.4.** Suppose $f : X \to Y$ is a constant function between metric spaces, say $f(x) = y_0$ for all $x \in X$. Show that $f$ is continuous.

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If you have any questions or feedback, please feel free to contact me via E-mail at hannah.kuempel@stat.uni-muenchen.de!!

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