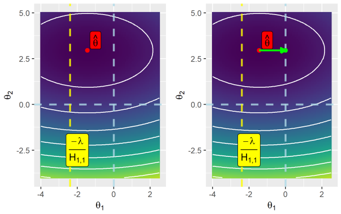
Introduction to Machine Learning

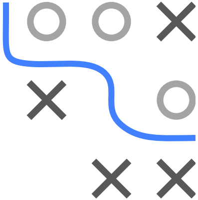
Regularization
Geometry of L1 Regularization



Learning goals

- Approximate transformation of unregularized minimizer to regularized
- Soft-Thresholding

L1-REGULARIZATION



- The L1-regularized risk of a model $f(\mathbf{x} \mid \boldsymbol{\theta})$ is

$$\mathcal{R}_{\text{reg}}(\boldsymbol{\theta}) = \mathcal{R}_{\text{emp}}(\boldsymbol{\theta}) + \sum_j \lambda |\theta_j|$$

  and the (sub-)gradient is:

$$\nabla_{\theta} \mathcal{R}_{\text{emp}}(\boldsymbol{\theta}) + \lambda \operatorname{sign}(\boldsymbol{\theta})$$

- Unlike in $L2$, contribution to grad. doesn't scale with θ_j elements.
- Again: quadratic Taylor approximation of $\mathcal{R}_{\text{emp}}(\boldsymbol{\theta})$ around its minimizer $\hat{\boldsymbol{\theta}}$, then regularize:

$$\tilde{\mathcal{R}}_{\text{reg}}(\boldsymbol{\theta}) = \mathcal{R}_{\text{emp}}(\hat{\boldsymbol{\theta}}) + \frac{1}{2}(\boldsymbol{\theta} - \hat{\boldsymbol{\theta}})^T \boldsymbol{H} (\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}) + \sum_j \lambda |\theta_j|$$

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- To cheat and simplify, we assume the $\boldsymbol{H}$ is diagonal, with $H_{j,j} \geq 0$
- Now $\tilde{\mathcal{R}}_{\text{reg}}(\boldsymbol{\theta})$ decomposes into sum over params θ_j (separable!):

$$\tilde{\mathcal{R}}_{\text{reg}}(\boldsymbol{\theta}) = \mathcal{R}_{\text{emp}}(\hat{\boldsymbol{\theta}}) + \sum_j \left[\frac{1}{2} H_{j,j}(\theta_j - \hat{\theta}_j)^2\right] + \sum_j \lambda |\theta_j|$$

- We can minimize analytically:

$$\begin{aligned}
\hat{\theta}_{\text{lasso},j} &= \text{sign}(\hat{\theta}_j) \max\left\{|\hat{\theta}_j| - \frac{\lambda}{H_{j,j}}, 0\right\} \\
&= \begin{cases}
\hat{\theta}_j + \frac{\lambda}{H_{j,j}} & \text{, if } \hat{\theta}_j < -\frac{\lambda}{H_{j,j}} \\
0 & \text{, if } \hat{\theta}_j \in [-\frac{\lambda}{H_{j,j}}, \frac{\lambda}{H_{j,j}}] \\
\hat{\theta}_j - \frac{\lambda}{H_{j,j}} & \text{, if } \hat{\theta}_j > \frac{\lambda}{H_{j,j}}
\end{cases}
\end{aligned}$$

- Shows how lasso (approx) transforms the normal minimizer
- If $H_{j,j} = 0$ exactly, $\hat{\boldsymbol{\theta}}_{\text{lasso},j} = 0$

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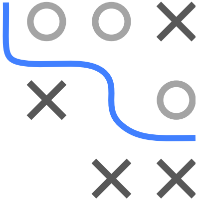
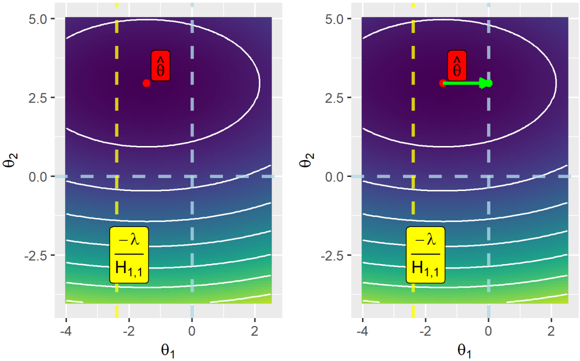


- If $0 < \hat{\theta}_j \leq \frac{\lambda}{H_{j,j}}$ or $0 > \hat{\theta}_j \geq -\frac{\lambda}{H_{j,j}}$, the optimal value of θ_j (for the regularized risk) is 0 because the contribution of $\mathcal{R}_{\text{emp}}(\boldsymbol{\theta})$ to $\mathcal{R}_{\text{reg}}(\boldsymbol{\theta})$ is overwhelmed by the L1 penalty, which forces it to be 0.



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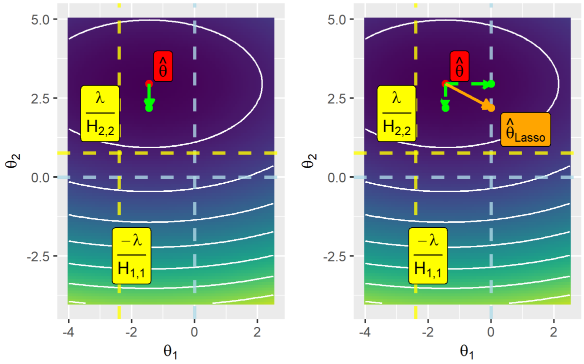
- If $0 < \frac{\lambda}{H_{j,j}} < \hat{\theta}_j$ or $0 > -\frac{\lambda}{H_{j,j}} > \hat{\theta}_j$, the $L1$ penalty shifts the optimal value of θ_j toward 0 by the amount $\frac{\lambda}{H_{j,j}}$.



- Yellow dotted lines are limits from soft-thresholding
- Therefore, the L1 penalty induces sparsity in the parameter vector.