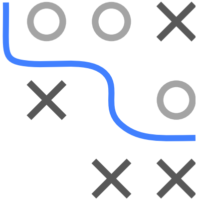
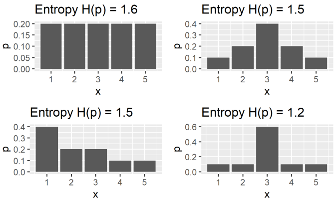
Introduction to Machine Learning

Information Theory
Joint Entropy and Mutual Information I





Learning goals

- Know the joint entropy
- Know conditional entropy as remaining uncertainty
- Know mutual information as the amount of information of an RV obtained by another

JOINT ENTROPY

- Recap: The **joint entropy** of two discrete RVs X and Y with joint pmf $p(x, y)$ is:

$$H(X, Y) = -\sum_{x \in \mathcal{X}} \sum_{y \in \mathcal{Y}} p(x, y) \log(p(x, y)),$$

  which can also be expressed as

$$H(X, Y) = -\mathbb{E}\left[\log(p(X, Y))\right].$$

- For continuous RVs X and Y with joint density $p(x, y)$, the differential joint entropy is:

$$h(X, Y) = -\int_{\mathcal{X} \times \mathcal{Y}} p(x, y) \log p(x, y) dxdy$$

For the rest of the section we will stick to the discrete case. Pretty much everything we show and discuss works in a completely analogous manner for the continuous case - if you change sums to integrals.

CONDITIONAL ENTROPY

- The **conditional entropy** $H(Y|X)$ quantifies the uncertainty of Y that remains if the outcome of X is given.
- $H(Y|X)$ is defined as the expected value of the entropies of the conditional distributions, averaged over the conditioning RV.
- If $(X, Y) \sim p(x, y)$, the conditional entropy $H(Y|X)$ is defined as

$$\begin{aligned} H(Y|X) = \mathbb{E}_X[H(Y|X=x)] &= \sum_{x \in \mathcal{X}} p(x) H(Y|X=x) \\ &= -\sum_{x \in \mathcal{X}} p(x) \sum_{y \in \mathcal{Y}} p(y|x) \log p(y|x) \\ &= -\sum_{x \in \mathcal{X}} \sum_{y \in \mathcal{Y}} p(x, y) \log p(y|x) \\ &= -\mathbb{E}\left[\log p(Y|X)\right]. \end{aligned}$$

- For the continuous case with density f we have

$$h(Y|X) = -\int f(x, y) \log f(x|y) dxdy.$$

CHAIN RULE FOR ENTROPY

The **chain rule for entropy** is analogous to the chain rule for probability and derives directly from it.

$$H(X, Y) = H(X) + H(Y|X)$$

Proof:
$$\begin{aligned}
H(X, Y) &= -\sum_{x \in \mathcal{X}} \sum_{y \in \mathcal{Y}} p(x, y) \log p(x, y) \\
&= -\sum_{x \in \mathcal{X}} \sum_{y \in \mathcal{Y}} p(x, y) \log p(x) p(y|x) \\
&= -\sum_{x \in \mathcal{X}} \sum_{y \in \mathcal{Y}} p(x, y) \log p(x) - \sum_{x \in \mathcal{X}} \sum_{y \in \mathcal{Y}} p(x, y) \log p(y|x) \\
&= -\sum_{x \in \mathcal{X}} p(x) \log p(x) - \sum_{x \in \mathcal{X}} \sum_{y \in \mathcal{Y}} p(x, y) \log p(y|x) \\
&= H(X) + H(Y|X)
\end{aligned}$$

n-variable version:

$$H(X_1, X_2, \ldots, X_n) = \sum_{i=1}^{n} H(X_i | X_{i-1}, \ldots, X_1).$$

JOINT AND CONDITIONAL ENTROPY

The following relations hold:

$$H(X, X) = H(X)$$
$$H(X|X) = 0$$
$$H((X, Y)|Z) = H(X|Z) + H(Y|(X, Z))$$

Which can all be trivially derived from the previous considerations.

Furthermore, if $H(X|Y) = 0$ and X, Y are discrete RV, then X is a function of Y, so for all y with $p(y) > 0$, there is only one x with $p(x, y) > 0$. Proof is not hard, but also not completely trivial.

MUTUAL INFORMATION

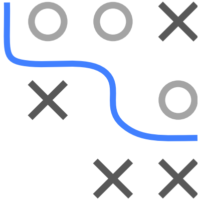
- The MI describes the amount of info about one RV obtained through another RV or how different their joint distribution is from pure independence.
- Consider two RVs X and Y with a joint pmf $p(x, y)$ and marginal pmfs $p(x)$ and $p(y)$. The MI $I(X; Y)$ is the Kullback-Leibler Divergence between the joint distribution and the product distribution $p(x)p(y)$:

$$\begin{aligned}
I(X;Y) &= \sum_{x \in \mathcal{X}} \sum_{y \in \mathcal{Y}} p(x,y) \log \frac{p(x,y)}{p(x)p(y)} \\
&= D_{KL}(p(x,y) \| p(x)p(y)) \\
&= \mathbb{E}_{p(x,y)} \left[\log \frac{p(X,Y)}{p(X)p(Y)} \right].
\end{aligned}$$

- For two continuous random variables with joint density $f(x, y)$:

$$I(X;Y) = \int f(x,y) \log \frac{f(x,y)}{f(x)f(y)} dxdy.$$

MUTUAL INFORMATION



We can rewrite the definition of mutual information $I(X;Y)$ as

$$
\begin{aligned}
I(X;Y) &= \sum_{x,y} p(x,y) \log \frac{p(x,y)}{p(x)p(y)} \\
&= \sum_{x,y} p(x,y) \log \frac{p(x|y)}{p(x)} \\
&= -\sum_{x,y} p(x,y) \log p(x) + \sum_{x,y} p(x,y) \log p(x|y) \\
&= -\sum_{x} p(x) \log p(x) - \left(-\sum_{x,y} p(x,y) \log p(x|y) \right) \\
&= H(X) - H(X|Y).
\end{aligned}
$$

So, $I(X;Y)$ is reduction in uncertainty of X due to knowledge of Y.

MUTUAL INFORMATION

The following relations hold:

$$I(X;Y) = H(X) - H(X|Y)$$
$$I(X;Y) = H(Y) - H(Y|X)$$
$$I(X;Y) \leq \min\{H(X), H(Y)\} \text{ if } X, Y \text{ are discrete RVs}$$
$$I(X;Y) = H(X) + H(Y) - H(X,Y)$$
$$I(X;Y) = I(Y;X)$$
$$I(X;X) = H(X)$$

All of the above are trivial to prove.

MUTUAL INFORMATION - EXAMPLE

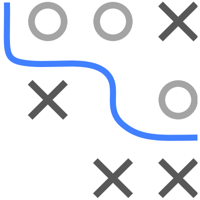
Let X, Y have the following joint distribution:

	X_1	X_2	X_3	X_4
Y_1	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{32}$
Y_2	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{32}$	$\frac{1}{32}$
Y_3	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$
Y_4	$\frac{1}{4}$	0	0	0

Marginal distribution of X is $(\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{8})$ and marginal distribution of Y is $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$, and hence $H(X) = \frac{7}{4}$ bits and $H(Y) = 2$ bits.

MUTUAL INFORMATION - EXAMPLE / 2

The conditional entropy $H(X|Y)$ is given by:



$$
\begin{aligned}
H(X|Y) &= \sum_{i=1}^{4} p(Y=i) H(X|Y=i) \\
&= \frac{1}{4} H\left(\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{8}\right) + \frac{1}{4} H\left(\frac{1}{4}, \frac{1}{2}, \frac{1}{8}, \frac{1}{8}\right) \\
&+ \frac{1}{4} H\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right) + \frac{1}{4} H(1, 0, 0, 0) \\
&= \frac{1}{4} \cdot \frac{7}{4} + \frac{1}{4} \cdot \frac{7}{4} + \frac{1}{4} \cdot 2 + \frac{1}{4} \cdot 0 \\
&= \frac{11}{8} \text{ bits.}
\end{aligned}
$$

Similarly, $H(Y|X) = \frac{13}{8}$ bits and $H(X, Y) = \frac{27}{8}$ bits.