SOFT-THRESHOLDING AND L1 REGULARIZATION / 2

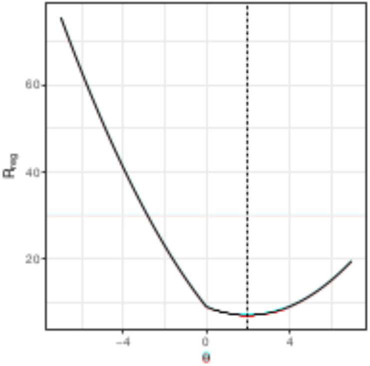
For strictly convex functions, there exists only one unique minimum and for convex functions a stationary point (if it exists) is a minimum.

We now separately investigate z_j for $\theta_j > 0$ and $\theta_j < 0$.

NB: on these halflines z_j is differentiable (with possible stationary point) since

- for $\theta_j > 0 : \frac{d}{d\theta_j}|\theta_j| = \frac{d}{d\theta_j}\theta_j = 1$,
- for $\theta_j < 0 : \frac{d}{d\theta_j}|\theta_j| = \frac{d}{d\theta_j}(-\theta_j) = -1$.

1) $\theta_j > 0$:



$$\frac{d}{d\theta_j} z_j(\theta_j) = H_{j,j}\theta_j - H_{j,j}\hat{\theta}_j + \lambda \overset{!}{=} 0$$

$$\Rightarrow \hat{\theta}_{\text{lasso},j} = \hat{\theta}_j - \frac{\lambda}{H_{j,j}} > 0$$

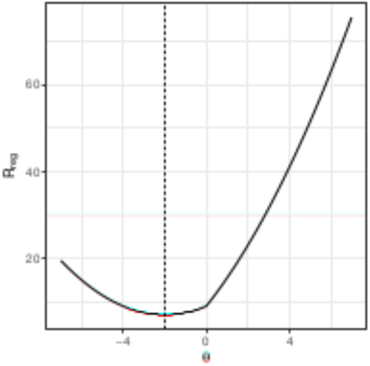
This minimum is only valid if $\hat{\theta}_{\text{lasso},j} > 0$ and so it must hold that

$$\hat{\theta}_j > \frac{\lambda}{H_{j,j}}.$$

SOFT-THRESHOLDING AND L1 REGULARIZATION

/ 4

2) $\hat{\theta}_{\text{lasso},j} < 0$:



$$\frac{d}{d\theta_j} z_j(\theta_j) = H_{j,j}\theta_j - H_{j,j}\hat{\theta}_j - \lambda \overset{!}{=} 0$$

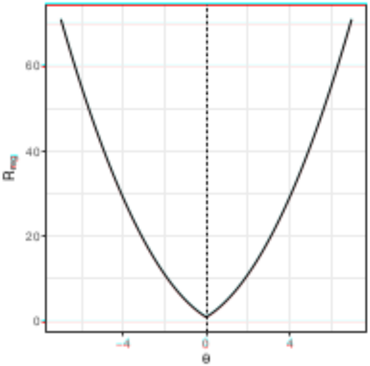
$$\Rightarrow \hat{\theta}_{\text{lasso},j} = \hat{\theta}_j + \frac{\lambda}{H_{j,j}} < 0$$

This minimum is only valid if $\hat{\theta}_{\text{lasso},j} < 0$ and so it must hold that

$$\hat{\theta}_j < -\frac{\lambda}{H_{j,j}}.$$

SOFT-THRESHOLDING AND L1 REGULARIZATION

/ 5



$\Rightarrow$ If $\hat{\theta}_j \in [-\frac{\lambda}{H_{j,j}}, \frac{\lambda}{H_{j,j}}]$ then z_j has no stationary point with

$$\hat{\theta}_{\text{lasso},j} < 0 \text{ or } \hat{\theta}_{\text{lasso},j} > 0.$$

However, a unique minimum must exist since z_j is strictly convex for $H_{j,j} > 0$. This means the only possible minimizer of z_j is $\hat{\theta}_{\text{lasso},j} = 0$.

$$\Rightarrow \hat{\theta}_{\text{lasso},j} = \begin{cases} \hat{\theta}_j + \frac{\lambda}{H_{j,j}} & \text{, if } \hat{\theta}_j < -\frac{\lambda}{H_{j,j}} \text{ and } H_{j,j} > 0 \\ 0 & \text{, if } \hat{\theta}_j \in [-\frac{\lambda}{H_{j,j}}, \frac{\lambda}{H_{j,j}}] \text{ or } H_{j,j} = 0 \\ \hat{\theta}_j - \frac{\lambda}{H_{j,j}} & \text{, if } \hat{\theta}_j > \frac{\lambda}{H_{j,j}} \text{ and } H_{j,j} > 0 \end{cases}$$