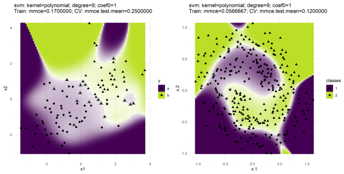
Introduction to Machine Learning

Nonlinear Support Vector Machines
The Polynomial Kernel



Learning goals

- Know the homogeneous and non-homogeneous polynomial kernel
- Understand the influence of the choice of the degree on the decision boundary

HOMOGENEOUS POLYNOMIAL KERNEL

$$k(\mathbf{x}, \tilde{\mathbf{x}}) = (\mathbf{x}^T \tilde{\mathbf{x}})^d, \text{ for } d \in \mathbb{N}$$

The feature map contains all monomials of exactly order d.

$$\phi(\mathbf{x}) = \left(\sqrt{\binom{d}{k_1, \ldots, k_p}} x_1^{k_1} \ldots x_p^{k_p} \right)_{k_i \geq 0, \sum_i k_i = d}$$

That $\langle \phi(\mathbf{x}), \phi(\tilde{\mathbf{x}}) \rangle = k(\mathbf{x}, \tilde{\mathbf{x}})$ holds can easily be checked by simple calculation and using the multinomial formula

$$(x_1 + \ldots + x_p)^d = \sum_{k_i \geq 0, \sum_i k_i = d} \binom{d}{k_1, \ldots, k_p} x_1^{k_1} \ldots x_p^{k_p}$$

The map $\phi(\mathbf{x})$ has $\binom{p+d-1}{d}$ dimensions. We see that $\phi(\mathbf{x})$ contains no terms of "lesser" order, so, e.g., linear effects. As an example for $p = d = 2$: $\phi(\mathbf{x}) = (x_1^2, x_2^2, \sqrt{2} x_1 x_2)$.

NONHOMOGENEOUS POLYNOMIAL KERNEL

$$k(\mathbf{x}, \tilde{\mathbf{x}}) = (\mathbf{x}^T \tilde{\mathbf{x}} + b)^d, \text{ for } b \geq 0, d \in \mathbb{N}$$

The maths is very similar as before, we kind of add a further constant term in the original space, with

$$(\mathbf{x}^T \tilde{\mathbf{x}} + b)^d = (x_1 \tilde{x}_1 + \ldots + x_p \tilde{x}_p + b)^d$$

The feature map contains all monomials up to order d.

$$\phi(\mathbf{x}) = \left(\sqrt{\binom{d}{k_1, \ldots, k_{p+1}}} x_1^{k_1} \ldots x_p^{k_p} b^{k_{p+1}/2} \right)_{k_i \geq 0, \sum_i k_i = d}$$

The map $\phi(\mathbf{x})$ has $\binom{p+d}{d}$ dimensions. For $p = d = 2$:

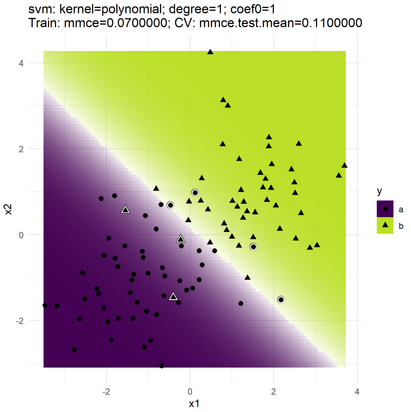
$$(x_1 \tilde{x}_1 + x_2 \tilde{x}_2 + b)^2 = x_1^2 \tilde{x}_1^2 + x_2^2 \tilde{x}_2^2 + 2 x_1 x_2 \tilde{x}_1 \tilde{x}_2 + 2b x_1 \tilde{x}_1 + 2b x_2 \tilde{x}_2 + b^2$$

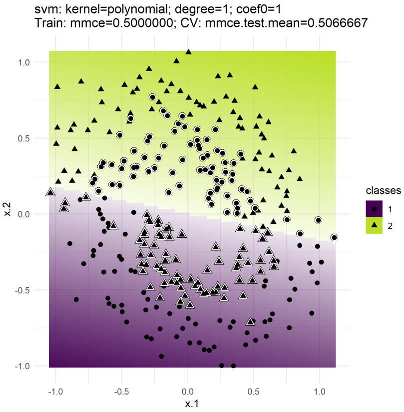
Therefore,

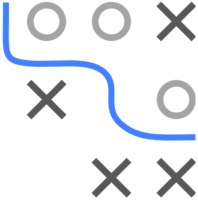
$$\phi(\mathbf{x}) = (x_1^2, x_2^2, \sqrt{2} x_1 x_2, \sqrt{2b} x_1, \sqrt{2b} x_2, b)$$

POLYNOMIAL KERNEL

Degree $d = 1$ yields a linear decision boundary.

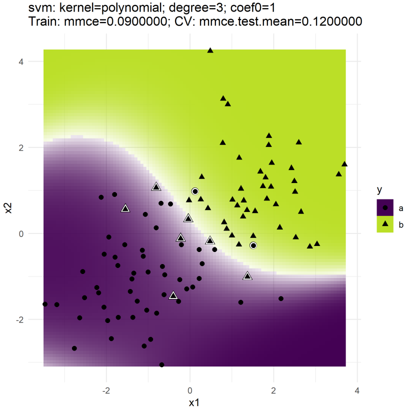


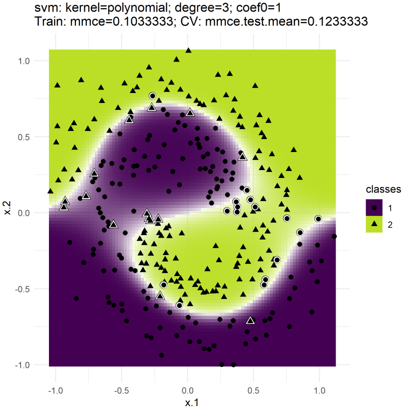


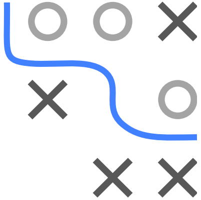


POLYNOMIAL KERNEL / 2

The higher the degree, the more nonlinearity in the decision boundary.

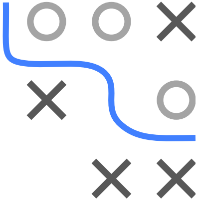


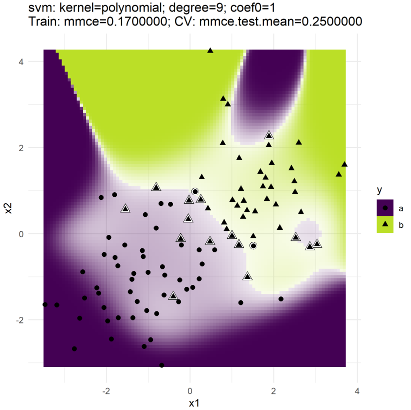


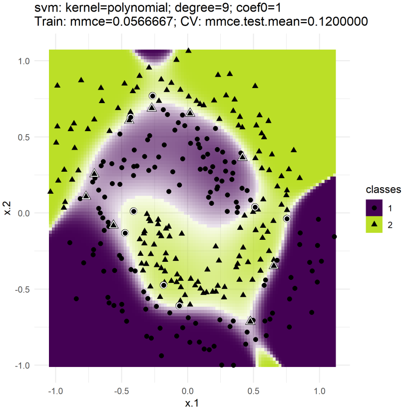


POLYNOMIAL KERNEL / 3

The higher the degree, the more nonlinearity in the decision boundary.







POLYNOMIAL KERNEL / 4

For $k(\mathbf{x}, \tilde{\mathbf{x}}) = (\mathbf{x}^\top \tilde{\mathbf{x}} + 0)^d$ we get no lower order effects.