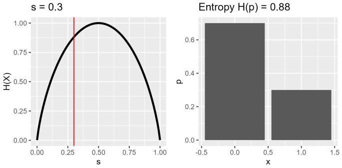
Introduction to Machine Learning

Information Theory
Entropy II



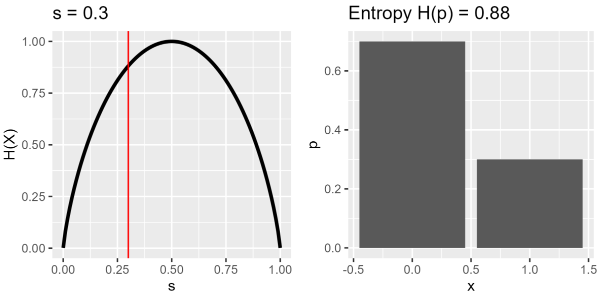
Learning goals

- Further properties of entropy and joint entropy
- Understand that uniqueness theorem justifies choice of entropy formula
- Maximum entropy principle

ENTROPY OF BERNOULLI DISTRIBUTION

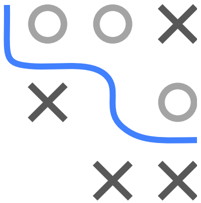
Let X be Bernoulli / a coin with $\mathbb{P}(X = 1) = s$ and $\mathbb{P}(X = 0) = 1 - s$.

$$H(X) = -s \cdot \log_2(s) - (1 - s) \cdot \log_2(1 - s).$$



We note: If the coin is deterministic, so $s = 1$ or $s = 0$, then $H(s) = 0$; $H(s)$ is maximal for $s = 0.5$, a fair coin. $H(s)$ increases monotonically the closer we get to $s = 0.5$. This all seems plausible.

JOINT ENTROPY



- The **joint entropy** of two discrete random variables X and Y is:

$$H(X, Y) = H(p_{X,Y}) = -\sum_{x \in \mathcal{X}} \sum_{y \in \mathcal{Y}} p(x, y) \log_2(p(x, y))$$

- Intuitively, the joint entropy is a measure of the total uncertainty in the two variables X and Y. In other words, it is simply the entropy of the joint distribution $p(x, y)$.
- There is nothing really new in this definition because $H(X, Y)$ can be considered to be a single vector-valued random variable.
- More generally:

$$H(X_1, X_2, \ldots, X_n) = -\sum_{x_1 \in \mathcal{X}_1} \cdots \sum_{x_n \in \mathcal{X}_n} p(x_1, x_2, \ldots, x_n) \log_2(p(x_1, x_2, \ldots, x_n))$$

ENTROPY IS ADDITIVE UNDER INDEPENDENCE

7. Entropy is additive for independent RVs.

Let X and Y be two independent RVs. Then:

$$
\begin{aligned}
H(X,Y) &= -\sum_{x\in\mathcal{X}}\sum_{y\in\mathcal{Y}} p(x,y)\log_2(p(x,y)) \\
&= -\sum_{x\in\mathcal{X}}\sum_{y\in\mathcal{Y}} p_X(x)p_Y(y)\log_2(p_X(x)p_Y(y)) \\
&= -\sum_{x\in\mathcal{X}}\sum_{y\in\mathcal{Y}} p_X(x)p_Y(y)\log_2(p_X(x)) + p_X(x)p_Y(y)\log_2(p_Y(y)) \\
&= -\sum_{x\in\mathcal{X}}\sum_{y\in\mathcal{Y}} p_X(x)p_Y(y)\log_2(p_X(x)) - \sum_{y\in\mathcal{Y}}\sum_{x\in\mathcal{X}} p_X(x)p_Y(y)\log_2(p_Y(y)) \\
&= -\sum_{x\in\mathcal{X}} p_X(x)\log_2(p_X(x)) - \sum_{y\in\mathcal{Y}} p_Y(y)\log_2(p_Y(y)) = H(X) + H(Y)
\end{aligned}
$$

THE UNIQUENESS THEOREM

▸ Khinchin 1957 showed that the only family of functions satisfying

- $H(p)$ is continuous in probabilities $p(x)$
- adding or removing an event with $p(x) = 0$ does not change it
- is additive for independent RVs
- is maximal for a uniform distribution.

is of the following form:

$$H(p) = -\lambda \sum_{x \in \mathcal{X}} p(x) \log p(x)$$

where λ is a positive constant. Setting $\lambda = 1$ and using the binary logarithm gives us the Shannon entropy.

THE MAXIMUM ENTROPY PRINCIPLE

Assume we know M properties about a discrete distribution $p(x)$ on $\mathcal{X}$, stated as "moment conditions" for functions $g_m(\cdot)$ and scalars α_m:

$$\mathbb{E}[g_m(X)] = \sum_{x \in \mathcal{X}} g_m(x)p(x) = \alpha_m \text{ for } m = 0, \ldots, M$$

Maximum entropy principle ▸ Jaynes 2003: Among all feasible distributions satisfying the constraints, choose the one with maximum entropy!

- Motivation: ensure no unwarranted assumptions on $p(x)$ are made beyond what we know.
- MEP follows similar logic to Occam's razor and principle of insufficient reason

THE MAXIMUM ENTROPY PRINCIPLE

Can be solved via Lagrangian multipliers (here with base e)

$$L(p(x),(\lambda_m)_{m=0}^M) = -\sum_{x\in\mathcal{X}} p(x)\log(p(x)) + \lambda_0\Big(\sum_{x\in\mathcal{X}} p(x) - 1\Big) + \sum_{m=1}^M \lambda_m\Big(\sum_{x\in\mathcal{X}} g_m(x)p(x) - \alpha_m\Big)$$

Finding critical points $p^*(x)$:

$$\frac{\partial L}{\partial p(x)} = -\log(p(x)) - 1 + \lambda_0 + \sum_{m=1}^M \lambda_m g_m(x) \overset{!}{=} 0 \iff p^*(x) = \exp(\lambda_0 - 1)\exp\Big(\sum_{m=1}^M \lambda_m g_m(x)\Big)$$

This is a maximum as $-1/p(x) < 0$. Since probs must sum to 1 we get

$$1 \overset{!}{=} \sum_{x\in\mathcal{X}} p^*(x) = \frac{1}{\exp(1-\lambda_0)} \sum_{x\in\mathcal{X}} \exp\Big(\sum_{m=1}^M \lambda_m g_m(x)\Big) \Rightarrow \exp(1-\lambda_0) = \sum_{x\in\mathcal{X}} \exp\Big(\sum_{m=1}^M \lambda_m g_m(x)\Big)$$

Plugging $\exp(1-\lambda_0)$ into $p^*(x)$ we obtain the constrained maxent distribution:

$$p^*(x) = \frac{\exp \sum_{m=1}^M \lambda_m g_m(x)}{\sum_{x\in\mathcal{X}} \exp \sum_{m=1}^M \lambda_m g_m(x)}$$

THE MAXIMUM ENTROPY PRINCIPLE

We now have: functional form of our distribution, up to M unknowns, the λ_m. But also: M equations, the moment conditions. So we can solve.

Example: Consider discrete RV representing a six-sided die roll and the moment condition $\mathbb{E}(X) = 4.8$. What is the maxent distribution?

- Condition means $g_1(x) = x$, $\alpha_1 = 4.8$. Then for some λ solution is

$$p^*(x) = \frac{\exp\left(\lambda g(x)\right)}{\sum_{j=1}^{6} \exp(\lambda g(x_j))} = \frac{\exp\left(\lambda x\right)}{\sum_{j=1}^{6} \exp\left(\lambda x_j\right)}$$

- Inserting into moment condition and solving (numerically) for λ:

$$4.8 \overset{!}{=} \sum_{j=1}^{6} x_j p^*(x_j) = \frac{e^{\lambda} + \ldots + 6(e^{\lambda})^6}{e^{\lambda} + \ldots + (e^{\lambda})^6} \Rightarrow \lambda \approx 0.5141$$

x	1	2	3	4	5	6
$p^*(x)$	3.22%	5.38%	9.01%	15.06%	25.19%	42.13%