

Introduction to Machine Learning

Information Theory

Differential Entropy



Learning goals

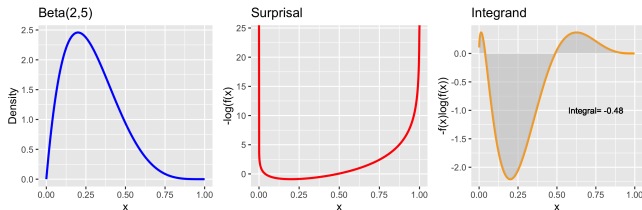
- Know that the entropy expresses expected information for continuous RVs
- Know the basic properties of the differential entropy

DIFFERENTIAL ENTROPY

- For a continuous random variable X with density function $f(x)$ and support \mathcal{X} , the analogue of entropy is **differential entropy**:

$$h(X) := h(f) := -\mathbb{E}[\log(f(x))] = - \int_{\mathcal{X}} f(x) \log(f(x)) dx$$

- The base of the log is again somewhat arbitrary, and we could either use 2 (and measure in bits) or e (to measure in nats).
- The integral above does not necessarily exist for all densities.
- Differential entropy lacks the non-negativeness of discrete entropy:
 $h(X) < 0$ is possible as $f(x) > 1$ is possible:



The diffent. is given
by the integral:
 $h(X) = -0.48$.



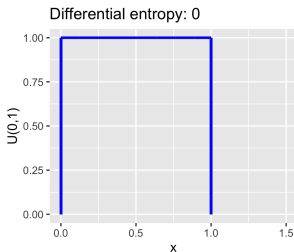
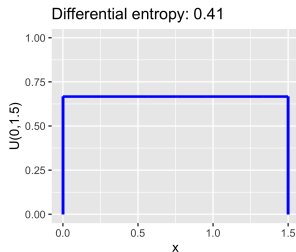
DIFF. ENTROPY OF UNIFORM DISTRIBUTION

Let X be a uniform random variable on $[0, a]$.

$$\begin{aligned}h(X) &= - \int_0^a f(x) \log(f(x)) dx \\&= - \int_0^a \frac{1}{a} \log\left(\frac{1}{a}\right) dx = \log(a)\end{aligned}$$



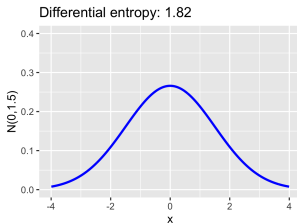
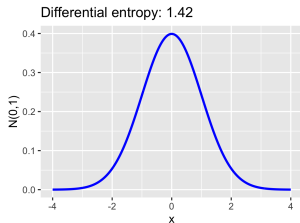
- For $a < 1$, $h(X) < 0. a$



DIFF. ENTROPY OF GAUSSIAN

Let $X \sim \mathcal{N}(\mu, \sigma^2)$ and let us measure in nats:

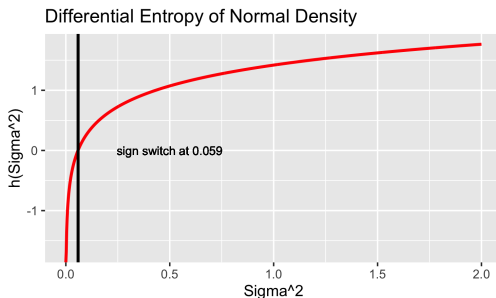
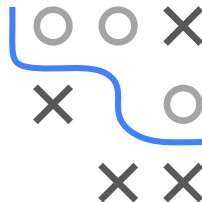
$$\begin{aligned}h(X) &= - \int_{\mathbb{R}} f(x) \log(f(x)) dx = - \int_{\mathbb{R}} f(x) \log \left(\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \right) dx \\&= - \int_{\mathbb{R}} f(x) \log \left(\frac{1}{\sqrt{2\pi\sigma^2}} \right) dx + \int_{\mathbb{R}} f(x) \frac{(x-\mu)^2}{2\sigma^2} dx \\&= - \log \left(\frac{1}{\sqrt{2\pi\sigma^2}} \right) \underbrace{\int_{\mathbb{R}} f(x) dx}_{=1} + \frac{1}{2\sigma^2} \underbrace{\int_{\mathbb{R}} f(x)(x-\mu)^2 dx}_{=:\sigma^2} \\&= \frac{1}{2} \log(2\pi\sigma^2) + \frac{1}{2} = \log(\sigma\sqrt{2\pi e})\end{aligned}$$



DIFF. ENTROPY OF GAUSSIAN

$$h(X) = - \int_{\mathbb{R}} f(x) \log(f(x)) dx = \log(\sigma \sqrt{2\pi e})$$

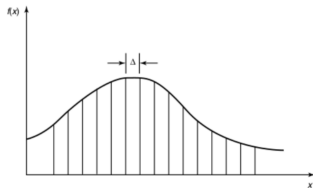
- $h(X)$ is not a function of μ (see translation invariance later).
- As σ^2 increases, the differential entropy also increases.
- For $\sigma^2 < \frac{1}{2\pi e} \approx 0.059$, it is negative.



DIFF. ENTROPY VS. DISCRETE

It is not so simple as to characterize $h(X)$ as a straightforward generalization of $H(X)$ of a limiting process. Consider the quantized random variable X^Δ , which is defined by

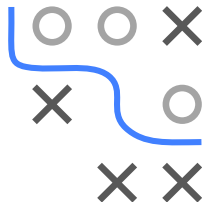
$$X^\Delta = x_i \quad \text{if} \quad i\Delta \leq X < (i+1)\Delta$$



If the density $f(x)$ of the random variable X is Riemann-integrable, then

$$H(X^\Delta) + \log(\Delta) \rightarrow h(X) \text{ as } \Delta \rightarrow 0.$$

Thus, the entropy of an n -bit quantization of a continuous random variable X is approximately $h(X) + n$.



JOINT DIFFERENTIAL ENTROPY

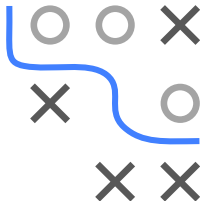
- For a continuous random vector X with density function $f(x)$ and support \mathcal{X} , differential entropy is also defined as:

$$h(X) = h(X_1, \dots, X_n) = h(f) = - \int_{\mathcal{X}} f(x) \log(f(x)) dx$$

- Hence this also defines the joint differential entropy for a set of continuous RVs.

Entropy of a multivariate normal distribution: If $X \sim N(\mu, \Sigma)$ is multivariate Gaussian, then

$$h(X) = \frac{1}{2} \log(2\pi e)^n |\Sigma| \quad (\text{nats})$$



PROPERTIES OF DIFFERENTIAL ENTROPY

- 1 $h(f)$ can be negative.
- 2 $h(f)$ is additive for independent RVs.
- 3 $h(f)$ is maximized by the multivariate normal, if we restrict to all distributions with the same (co)variance, so
$$h(X) \leq \frac{1}{2} \log(2\pi e)^n |\Sigma|.$$
- 4 $h(f)$ is maximized by the continuous uniform distribution for a random variable with a fixed range.
- 5 Translation-invariant, $h(X + a) = h(X)$.
- 6 $h(aX) = h(X) + \log |a|$.
- 7 $h(AX) = h(X) + \log |A|$ for random vectors and matrix A .



3) and 4) are slightly involved to prove, while the other properties are relatively straightforward to show