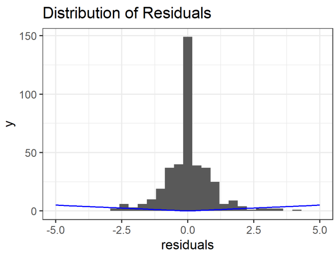
Introduction to Machine Learning

Advanced Risk Minimization
Maximum Likelihood Estimation vs. Empirical Risk Minimization



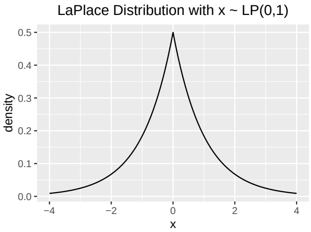
Learning goals

- Correspondence between Laplace errors and L1 loss
- Correspondence between Bernoulli targets and the Bernoulli / log loss

LAPLACE ERRORS - L1-LOSS

Let's consider Laplacian errors ϵ now, with density:

$$\frac{1}{2\sigma} \exp\left(-\frac{|\epsilon|}{\sigma}\right), \sigma > 0.$$



Then

$$y = f_{\text{true}}(\mathbf{x}) + \epsilon$$

also follows Laplace distrib. with mean $f\left(\mathbf{x}^{(i)} \mid \boldsymbol{\theta}\right)$ and scale σ.

LAPLACE ERRORS - L1-LOSS / 2

The likelihood is then

$$\begin{aligned}\mathcal{L}(\boldsymbol{\theta}) &= \prod_{i=1}^{n} p\left(y^{(i)} \,\middle|\, f\left(\mathbf{x}^{(i)} \mid \boldsymbol{\theta}\right), \sigma\right) \\ &\propto \exp\left(-\frac{1}{\sigma}\sum_{i=1}^{n}\left|y^{(i)} - f\left(\mathbf{x}^{(i)} \mid \boldsymbol{\theta}\right)\right|\right).\end{aligned}$$

The negative log-likelihood is

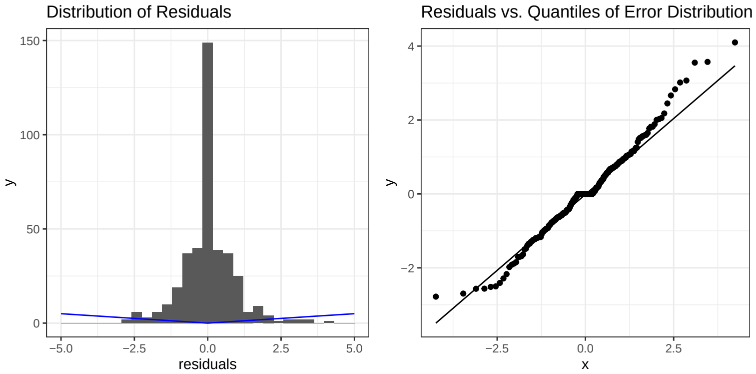
$$-\ell(\boldsymbol{\theta}) \propto \sum_{i=1}^{n}\left|y^{(i)} - f\left(\mathbf{x}^{(i)} \mid \boldsymbol{\theta}\right)\right|.$$

MLE for Laplacian errors = ERM with L1-loss.

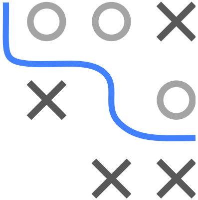
- Some losses correspond to more complex or less known error densities, like the Huber loss ▸ Meyer 2021
- Huber density is (unsurprisingly) a hybrid of Gaussian and Laplace

LAPLACE ERRORS - L1-LOSS / 3

- We simulate data $y \mid \mathbf{x} \sim \text{Laplacian}\left(f_{\text{true}}(\mathbf{x}), 1\right)$ with $f_{\text{true}} = 0.2 \cdot \mathbf{x}$.
- We can plot the empirical error distribution, i.e. the distribution of the residuals after fitting a regression model w.r.t. $L1$-loss.
- With the help of a Q-Q-plot we can compare the empirical residuals vs. the theoretical quantiles of a Laplacian distribution.



MAXIMUM LIKELIHOOD IN CLASSIFICATION



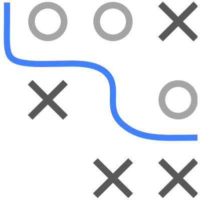
Let us assume the outputs y to be Bernoulli-distributed, i.e.

$$y \mid \mathbf{x} \sim \text{Ber}(\pi_{\text{true}}(\mathbf{x})).$$

The negative log likelihood is

$$\begin{aligned}
-\ell(\boldsymbol{\theta}) &= -\sum_{i=1}^{n} \log p\left(y^{(i)} \mid \mathbf{x}^{(i)}, \boldsymbol{\theta}\right) \\
&= -\sum_{i=1}^{n} \log \left[\pi\left(\mathbf{x}^{(i)}\right)^{y^{(i)}} \cdot \left(1 - \pi\left(\mathbf{x}^{(i)}\right)\right)^{(1-y^{(i)})}\right] \\
&= \sum_{i=1}^{n} -y^{(i)} \log[\pi\left(\mathbf{x}^{(i)}\right)] - \left(1 - y^{(i)}\right) \log[1 - \pi\left(\mathbf{x}^{(i)}\right)].
\end{aligned}$$

MAXIMUM LIKELIHOOD IN CLASSIFICATION / 2



This gives rise to the following loss function

$$L(y, \pi(\mathbf{x})) = -y \log(\pi(\mathbf{x})) - (1-y) \log(1-\pi(\mathbf{x})), \quad y \in \{0, 1\}$$

which we introduced as **Bernoulli** loss.

MAXIMUM LIKELIHOOD IN CLASSIFICATION / 3