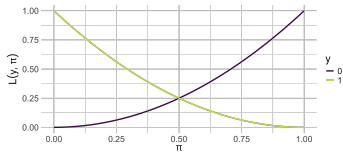
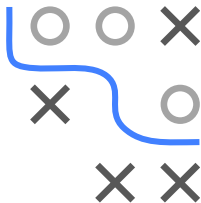


Introduction to Machine Learning

Advanced Risk Minimization

Brier Score



Learning goals

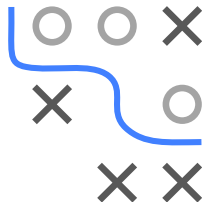
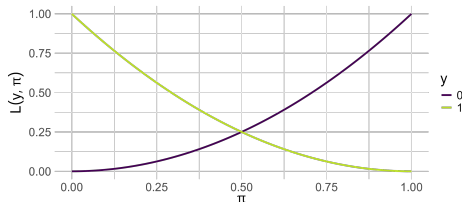
- Know the Brier score
- Derive the risk minimizer
- Derive the optimal constant model

BRIER SCORE

The binary Brier score is defined on probabilities $\pi \in [0, 1]$ and 0-1-encoded labels $y \in \{0, 1\}$ and measures their squared distance ($L2$ loss on probabilities).

$$L(y, \pi) = (\pi - y)^2$$

As the Brier score is a proper scoring rule, it can be used for calibration. Note that it is not convex on probabilities anymore.



BRIER SCORE: RISK MINIMIZER

The risk minimizer for the (binary) Brier score is

$$\pi^*(\mathbf{x}) = \eta(\mathbf{x}) = \mathbb{P}(y = 1 \mid \mathbf{x} = \mathbf{x}),$$

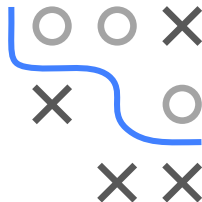
which means that the Brier score attains its minimum if the prediction equals the “true” probability of the outcome.

The risk minimizer for the multiclass Brier score is

$$\pi^*(\mathbf{x}) = \mathbb{P}(y = k \mid \mathbf{x} = \mathbf{x}).$$

Proof: We only show the proof for the binary case. We need to minimize

$$\mathbb{E}_{\mathbf{x}} [L(1, \pi(\mathbf{x})) \cdot \eta(\mathbf{x}) + L(0, \pi(\mathbf{x})) \cdot (1 - \eta(\mathbf{x}))],$$



BRIER SCORE: OPTIMAL CONSTANT MODEL

The optimal constant probability model $\pi(\mathbf{x}) = \theta$ w.r.t. the Brier score for labels from $\mathcal{Y} = \{0, 1\}$ is:

$$\begin{aligned}\min_{\theta} \mathcal{R}_{\text{emp}}(\theta) &= \min_{\theta} \sum_{i=1}^n \left(y^{(i)} - \theta\right)^2 \\ \Leftrightarrow \frac{\partial \mathcal{R}_{\text{emp}}(\theta)}{\partial \theta} &= -2 \cdot \sum_{i=1}^n (y^{(i)} - \theta) = 0 \\ \hat{\theta} &= \frac{1}{n} \sum_{i=1}^n y^{(i)}.\end{aligned}$$

This is the fraction of class-1 observations in the observed data.
(This also directly follows from our $L2$ proof for regression).

Similarly, for the multiclass brier score the optimal constant is

$$\hat{\theta}_k = \frac{1}{n} \sum_{i=1}^n [y = k].$$

