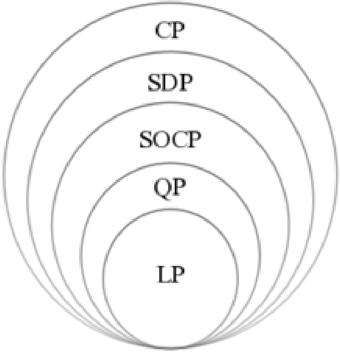
Optimization in Machine Learning

Algorithms for linear programs



Learning goals

- Definition
- Max. Likelihood
- Normal regression
- Risk Minimization

SIMPLEX ALGORITHM

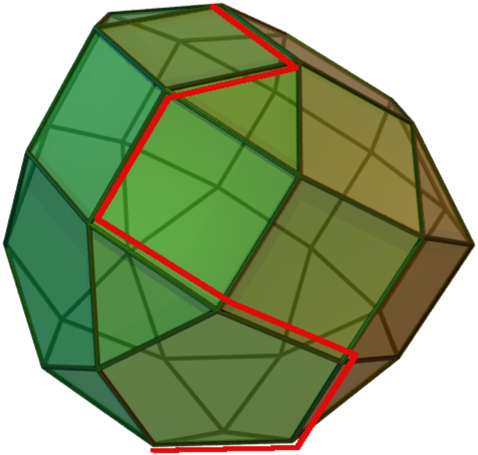
The Simplex algorithm is the most important method for solving Linear programming. It was published in 1947 by Georg Dantzig.

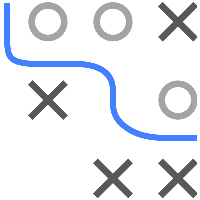
Basic idea: start from an arbitrary corner of the polytope. Run along this edge as long as the solution improves. Find a new edge, ...

Output: a path along the corners of the polytope that ends at the optimal point of the polytope.

Since linear programming is a **convex** optimization problem, the optimal corner found in this way is also a global optimum.

SIMPLEX ALGORITHM / 2





SIMPLEX ALGORITHM / 3

The simplex algorithm can be divided into two steps:

- **Phase I:** determination of a **starting point**
- **Phase II:** determination of the **optimal solution**

To be able to start, a starting point must first be found in **Phase I**, i.e. a feasible corner $\boldsymbol{x}_0$.

In **phase II** this solution is iteratively improved by searching for an edge that improves the solution and running along it to the next corner.

SIMPLEX ALGORITHM / 4

Phase I:
One way to find a starting point $\boldsymbol{x}_0$ is to solve a auxiliary linear problem with artificial variables $\boldsymbol{\epsilon}$:

$$\begin{aligned} \min_{\epsilon_1,\ldots,\epsilon_m} \quad & \sum_{i=1}^{m} \epsilon_i \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} + \boldsymbol{\epsilon} \geq \boldsymbol{b} \\ & \epsilon_1, \ldots, \epsilon_m \geq 0 \\ & \boldsymbol{x} \geq 0 \end{aligned}$$

- A feasible starting point for the auxiliary problem is $\boldsymbol{x} = \boldsymbol{0}$ and
  $\epsilon_i = \begin{cases} 0 & \text{if } b_i < 0 \\ b_i & \text{if } b_i \geq 0 \end{cases}$

SIMPLEX ALGORITHM / 5

- We then apply phase II of the simplex algorithm to the auxiliary problem.
- If the original problem has a feasible solution, then the optimal solution of the auxiliary problem **must** be $\epsilon = (0, ..., 0)$ (all artificial variables disappear) and the objective function is 0.
- If we find a solution with $\epsilon = \mathbf{0}$, then we have found a valid starting point.
- If we do not find a solution with $\epsilon = \mathbf{0}$, the problem can not be solved.

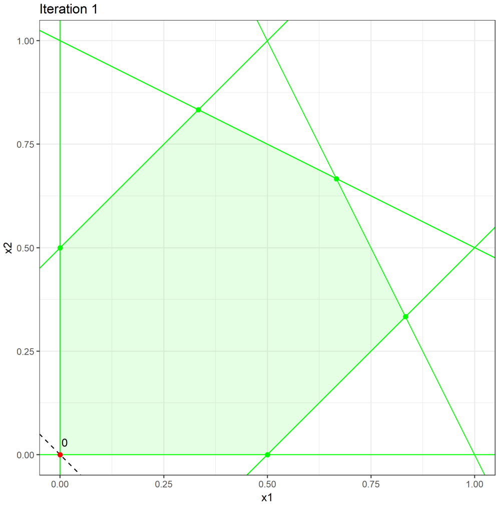
SIMPLEX ALGORITHM / 6

Example:

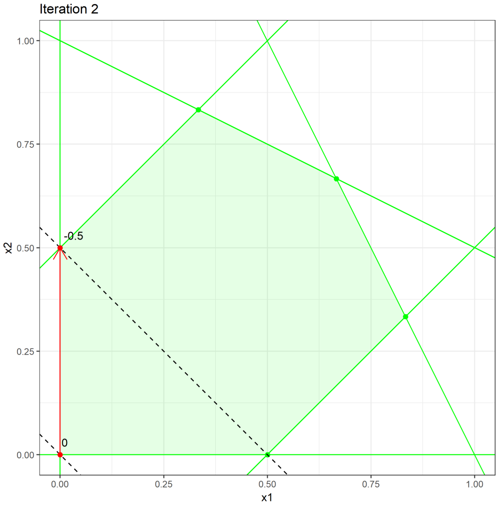
$$
\begin{aligned}
\min_{\boldsymbol{x}\in\mathbb{R}^2} \quad & -x_1 - x_2 \\
\text{s.t.} \quad & x_1 - x_2 \geq -0.5 \\
& -x_1 - 2x_2 \geq -2 \\
& -2x_1 - x_2 \geq -2 \\
& -x_1 + x_2 \geq -0.5 \\
& \boldsymbol{x} \geq 0
\end{aligned}
$$

A starting point is the corner $(\mathbf{0}, \mathbf{0})$.

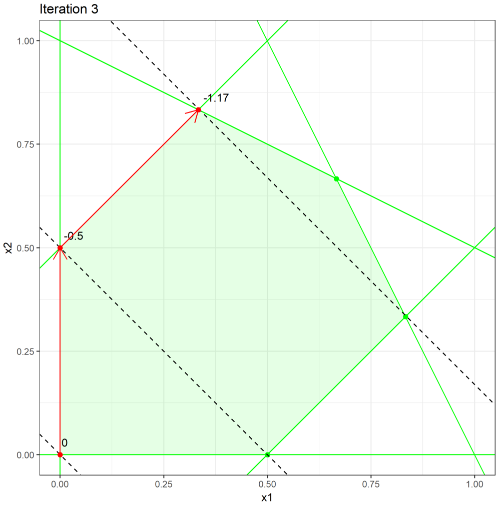
SIMPLEX ALGORITHM / 7

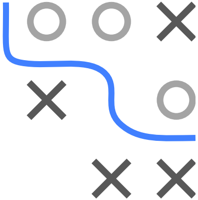


SIMPLEX ALGORITHM / 8



SIMPLEX ALGORITHM / 9





SIMPLEX ALGORITHM / 10