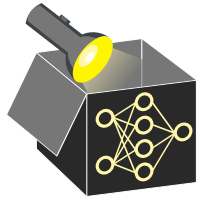
# Interpretable Machine Learning



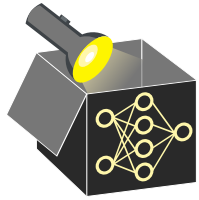
## Shapley Values


**Learning goals**

- Learn what game theory is
- Understand the concept behind cooperative games
- Understand the Shapley value in game theory

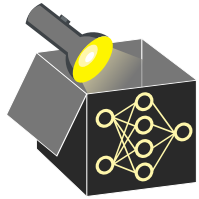
# COOPERATIVE GAMES IN GAME THEORY ▸ Shapley (1951)

- Game theory is the study of strategic games between players, “game” refers to any series of interactions between actors/agents with gains and losses of quantifiable utility value



# COOPERATIVE GAMES IN GAME THEORY

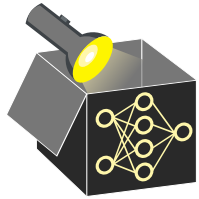
▸ Shapley (1951)



- Game theory is the study of strategic games between players, “game” refers to any series of interactions between actors/agents with gains and losses of quantifiable utility value
- Cooperative games: For all possible players $P = \{1, \ldots, p\}$, each subset of players $S \subseteq P$ forms a coalition – each coalition $S$ achieves a certain payout

# COOPERATIVE GAMES IN GAME THEORY

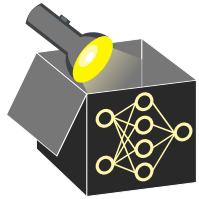
▸ Shapley (1951)



- Game theory is the study of strategic games between players, “game” refers to any series of interactions between actors/agents with gains and losses of quantifiable utility value
- Cooperative games: For all possible players $P = \{1, \ldots, p\}$, each subset of players $S \subseteq P$ forms a coalition – each coalition $S$ achieves a certain payout
- A value function $v : 2^P \mapsto \mathbb{R}$ maps all $2^{|P|}$ possible coalitions to their payout (or gain)

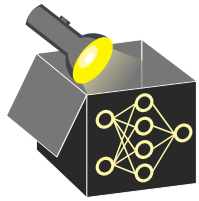
# COOPERATIVE GAMES IN GAME THEORY

▸ Shapley (1951)



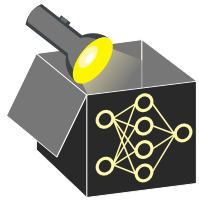
- Game theory is the study of strategic games between players, “game” refers to any series of interactions between actors/agents with gains and losses of quantifiable utility value
- Cooperative games: For all possible players $P = \{1, \ldots, p\}$, each subset of players $S \subseteq P$ forms a coalition – each coalition $S$ achieves a certain payout
- A value function $v : 2^P \mapsto \mathbb{R}$ maps all $2^{|P|}$ possible coalitions to their payout (or gain)
- $v(S)$ is the payout of coalition $S \subseteq P$ (payout of empty coalition must be zero: $v(\emptyset) = 0$)

# COOPERATIVE GAMES IN GAME THEORY ▸ Shapley (1951)



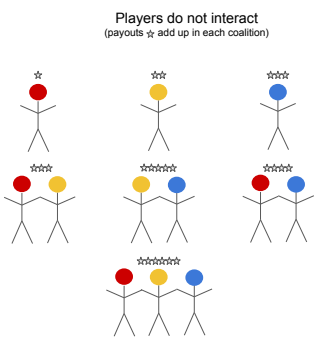
- Game theory is the study of strategic games between players, “game” refers to any series of interactions between actors/agents with gains and losses of quantifiable utility value
- Cooperative games: For all possible players $P = \{1, \ldots, p\}$, each subset of players $S \subseteq P$ forms a coalition – each coalition $S$ achieves a certain payout
- A value function $v : 2^P \mapsto \mathbb{R}$ maps all $2^{|P|}$ possible coalitions to their payout (or gain)
- $v(S)$ is the payout of coalition $S \subseteq P$ (payout of empty coalition must be zero: $v(\emptyset) = 0$)
- As some players contribute more than others, we want to fairly divide the total achievable payout $v(P)$ among the players according to a player’s individual contribution

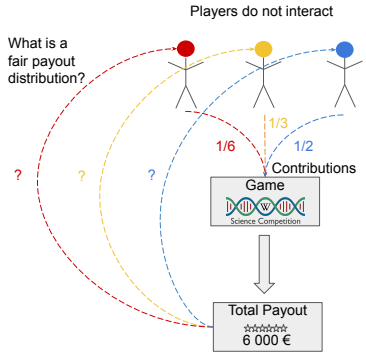
# COOPERATIVE GAMES IN GAME THEORY ▸ Shapley (1951)

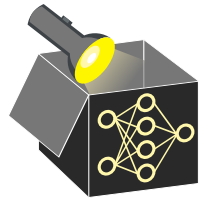


- Game theory is the study of strategic games between players, “game” refers to any series of interactions between actors/agents with gains and losses of quantifiable utility value
- Cooperative games: For all possible players $P = \{1, \ldots, p\}$, each subset of players $S \subseteq P$ forms a coalition – each coalition $S$ achieves a certain payout
- A value function $v : 2^P \mapsto \mathbb{R}$ maps all $2^{|P|}$ possible coalitions to their payout (or gain)
- $v(S)$ is the payout of coalition $S \subseteq P$ (payout of empty coalition must be zero: $v(\emptyset) = 0$)
- As some players contribute more than others, we want to fairly divide the total achievable payout $v(P)$ among the players according to a player’s individual contribution
- We call the individual payout per player $\phi_j$, $j \in P$ (later: Shapley value)

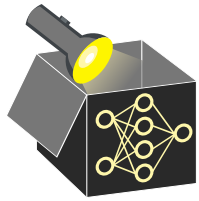
# COOPERATIVE GAMES WITHOUT INTERACTIONS

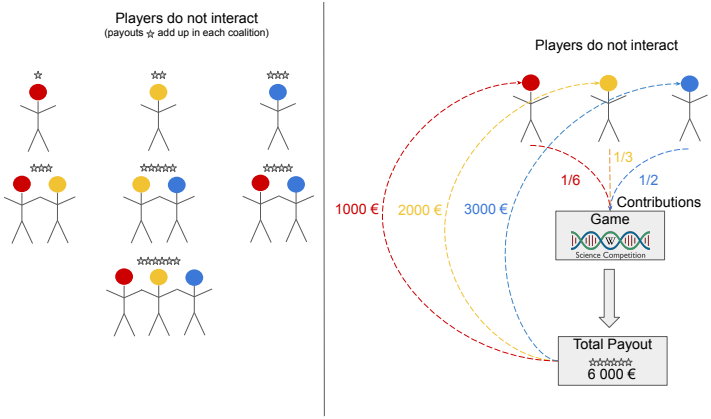


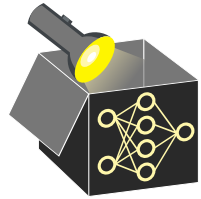
# COOPERATIVE GAMES WITHOUT INTERACTIONS

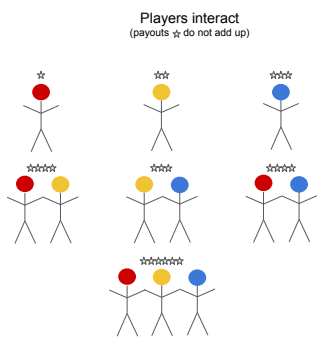


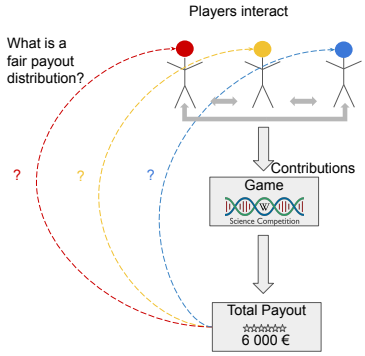



$\Rightarrow$ Fair Payouts are Trivial Without Interactions

# COOPERATIVE GAMES WITH INTERACTIONS

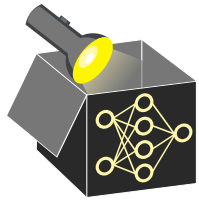




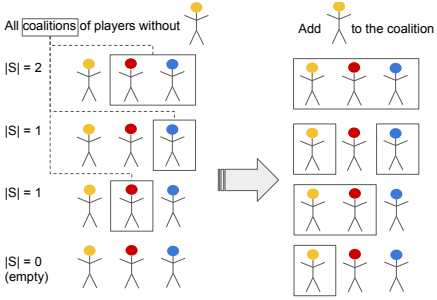




⇒ Unclear how to fairly distribute payouts when players interact

# COOPERATIVE GAMES WITH INTERACTIONS

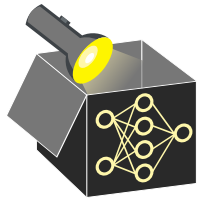


**Question:** What is a fair payout for player “yellow”?
**Idea:** Compute marginal contribution of the player of interest across different coalitions

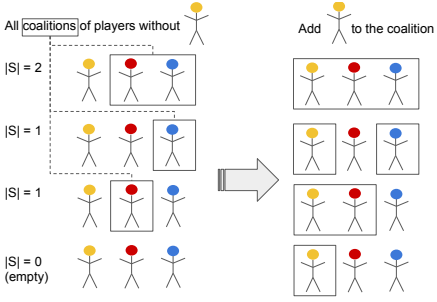



- Compute the total payout of each coalition
- Compute difference in payouts for each coalition with and without player “yellow” (= marginal contribution)
- Average marginal contributions using appropriate weights

# COOPERATIVE GAMES WITH INTERACTIONS

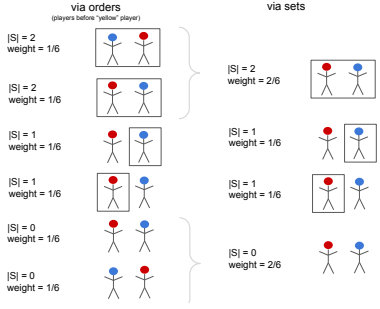


**Question:** What is a fair payout for player “yellow”?
**Idea:** Compute marginal contribution of the player of interest across different coalitions

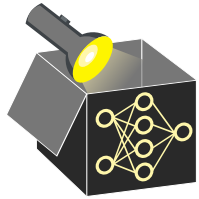



- Compute the total payout of each coalition
- Compute difference in payouts for each coalition with and without player “yellow” (= marginal contribution)
- Average marginal contributions using appropriate weights

**Note:** Each marginal contribution is weighted w.r.t. number of possible orders of its coalition
$\rightsquigarrow$ More players in $S \Rightarrow$ more orderings of $S$



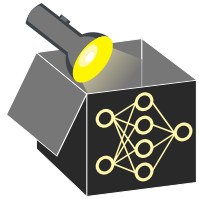
# SHAPLEY VALUE - SET DEFINITION



This idea refers to the **Shapley value** which assigns a payout value to each player according to its marginal contribution in all possible coalitions.

- Let $v(S \cup \{j\}) - v(S)$ be the marginal contribution of player $j$ to coalition $S$
  $\rightsquigarrow$ measures how much a player $j$ increases the value of a coalition $S$

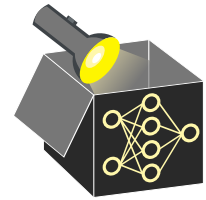
# SHAPLEY VALUE - SET DEFINITION



This idea refers to the **Shapley value** which assigns a payout value to each player according to its marginal contribution in all possible coalitions.

- Let $v(S \cup \{j\}) - v(S)$ be the marginal contribution of player $j$ to coalition $S$
  $\rightsquigarrow$ measures how much a player $j$ increases the value of a coalition $S$
- Average marginal contributions for all possible coalitions $S \subseteq P \setminus \{j\}$
  $\rightsquigarrow$ order of how players join the coalition matters $\Rightarrow$ different weights depending on size of $S$

# SHAPLEY VALUE - SET DEFINITION

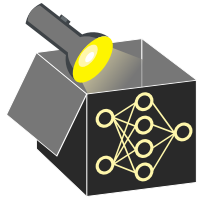


This idea refers to the **Shapley value** which assigns a payout value to each player according to its marginal contribution in all possible coalitions.

- Let $v(S \cup \{j\}) - v(S)$ be the marginal contribution of player $j$ to coalition $S$
  $\rightsquigarrow$ measures how much a player $j$ increases the value of a coalition $S$
- Average marginal contributions for all possible coalitions $S \subseteq P \setminus \{j\}$
  $\rightsquigarrow$ order of how players join the coalition matters $\Rightarrow$ different weights depending on size of $S$
- Shapley value via **set definition** (weighting via multinomial coefficient):

$$\phi_j = \sum_{S \subseteq P \setminus \{j\}} \frac{|S|!(|P| - |S| - 1)!}{|P|!} (v(S \cup \{j\}) - v(S))$$

# SHAPLEY VALUE - ORDER DEFINITION

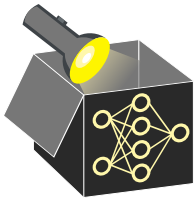


The Shapley value was introduced as summation over sets $S \subseteq P \setminus \{j\}$, but it can be equivalently defined as a summation of all orders of players:

$$\phi_j = \frac{1}{|P|!} \sum_{\tau \in \Pi} (v(S_j^\tau \cup \{j\}) - v(S_j^\tau))$$

- $\Pi$: All possible orders of players (we have $|P|!$ in total)

# SHAPLEY VALUE - ORDER DEFINITION



The Shapley value was introduced as summation over sets $S \subseteq P \setminus \{j\}$, but it can be equivalently defined as a summation of all orders of players:

$$\phi_j = \frac{1}{|P|!} \sum_{\tau \in \Pi} (v(S_j^\tau \cup \{j\}) - v(S_j^\tau))$$

- $\Pi$: All possible orders of players (we have $|P|!$ in total)
- $S_j^\tau$: Set of players before player $j$ in order $\tau = (\tau^{(1)}, \ldots, \tau^{(p)})$ where $\tau^{(i)}$ is $i$-th element

  $\Rightarrow$ Example: Players $1, 2, 3$ $\Rightarrow$

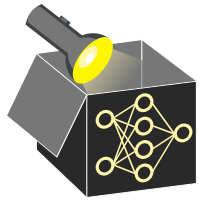
  $\Pi = \{(1,2,3), (1,3,2), (2,1,3), (2,3,1), (3,1,2), (3,2,1)\}$

  $\rightsquigarrow$ For order $\tau = (2,1,3)$ and player of interest $j = 3$ $\Rightarrow$ $S_j^\tau = \{2,1\}$

  $\rightsquigarrow$ For order $\tau = (3,1,2)$ and player of interest $j = 1$ $\Rightarrow$ $S_j^\tau = \{3\}$

  $\rightsquigarrow$ For order $\tau = (3,1,2)$ and player of interest $j = 3$ $\Rightarrow$ $S_j^\tau = \emptyset$

# SHAPLEY VALUE - ORDER DEFINITION

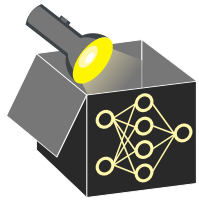


The Shapley value was introduced as summation over sets $S \subseteq P \setminus \{j\}$, but it can be equivalently defined as a summation of all orders of players:

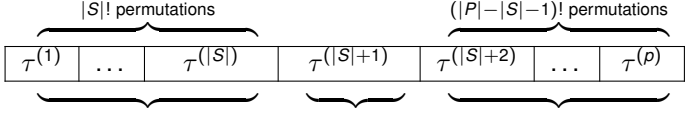
$$\phi_j = \frac{1}{|P|!} \sum_{\tau \in \Pi} \left( v(S_j^\tau \cup \{j\}) - v(S_j^\tau) \right)$$

- $\Pi$: All possible orders of players (we have $|P|!$ in total)
- $S_j^\tau$: Set of players before player $j$ in order $\tau = (\tau^{(1)}, \ldots, \tau^{(p)})$ where $\tau^{(i)}$ is $i$-th element
  $\Rightarrow$ Example: Players $1, 2, 3 \Rightarrow$
  $\Pi = \{(1,2,3), (1,3,2), (2,1,3), (2,3,1), (3,1,2), (3,2,1)\}$
  - $\rightsquigarrow$ For order $\tau = (2,1,3)$ and player of interest $j = 3 \Rightarrow S_j^\tau = \{2,1\}$
  - $\rightsquigarrow$ For order $\tau = (3,1,2)$ and player of interest $j = 1 \Rightarrow S_j^\tau = \{3\}$
  - $\rightsquigarrow$ For order $\tau = (3,1,2)$ and player of interest $j = 3 \Rightarrow S_j^\tau = \emptyset$
- Order definition: Marginal contribution of orders that yield set $S = \{1,2\}$ is summed twice
  $\rightsquigarrow$ In set definition, it has the weight $\frac{2!(3-2-1)!}{3!} = \frac{2 \cdot 0!}{6} = \frac{2}{6}$

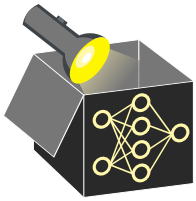
# SHAPLEY VALUE - COMMENTS ON ORDER DEFINITION



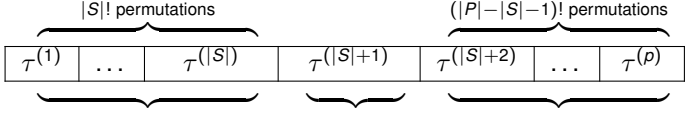
- Order and set definition are equivalent
- Reason: The number of orders which yield the same coalition $S$ is $|S|!(|P| - |S| - 1)!$
  $\Rightarrow$ There are $|S|!$ possible orders of players within coalition $S$
  $\Rightarrow$ There are $(|P| - |S| - 1)!$ possible orders of players without $S$ and $j$



# SHAPLEY VALUE - COMMENTS ON ORDER DEFINITION



- Order and set definition are equivalent
- Reason: The number of orders which yield the same coalition $S$ is $|S|!(|P| - |S| - 1)!$
  $\Rightarrow$ There are $|S|!$ possible orders of players within coalition $S$
  $\Rightarrow$ There are $(|P| - |S| - 1)!$ possible orders of players without $S$ and $j$

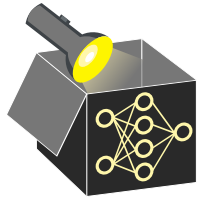



- Relevance of the order definition: Approximate Shapley values by sampling permutations
  $\rightsquigarrow$ randomly sample a fixed number of $M$ permutations and average them:

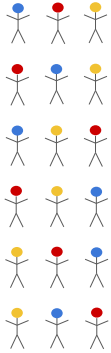
$$\phi_j = \frac{1}{M} \sum_{\tau \in \Pi_M} \left( v(S_j^\tau \cup \{j\}) - v(S_j^\tau) \right)$$

where $\Pi_M \subset \Pi$ is a random subset of $\Pi$ containing only $M$ orders of players

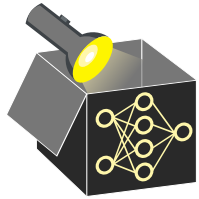
# WEIGHTS FOR MARGINAL CONTRIBUTION - ILLUSTRATION

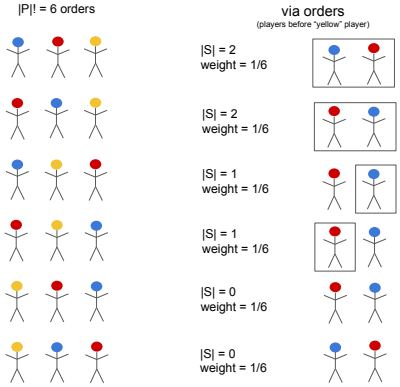


|P|! = 6 orders

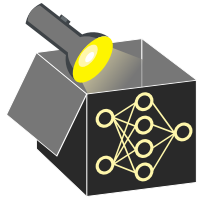


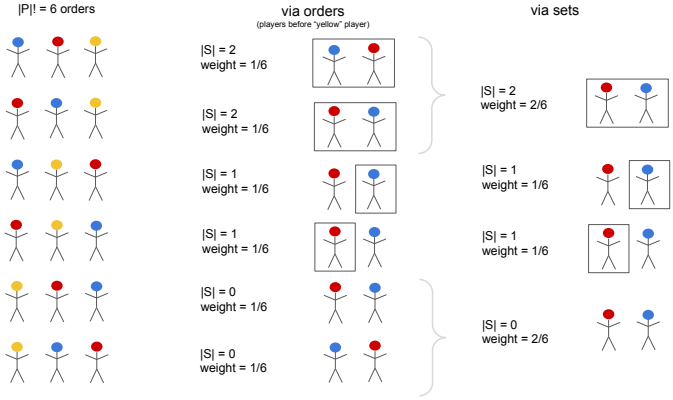
# WEIGHTS FOR MARGINAL CONTRIBUTION - ILLUSTRATION



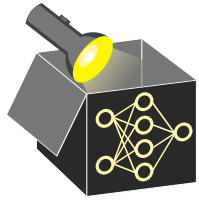


# WEIGHTS FOR MARGINAL CONTRIBUTION - ILLUSTRATION

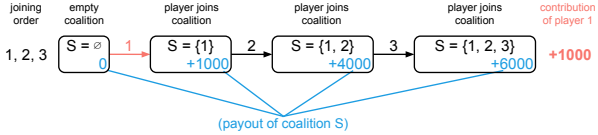




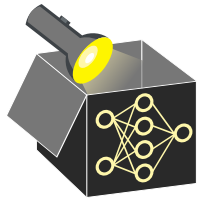
# SHAPLEY VALUES - ILLUSTRATION



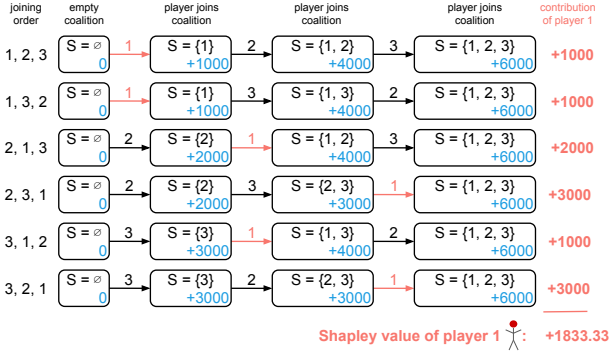
- Shapley value of player *j* is the marginal contribution to the value when it enters any coalition
- Produce all possible joining orders of player coalitions



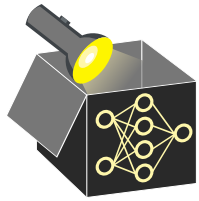
# SHAPLEY VALUES - ILLUSTRATION



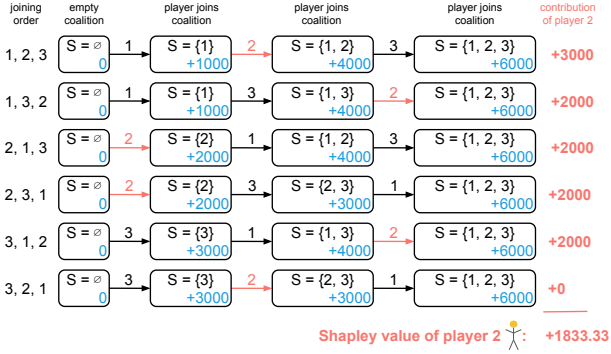
- Shapley value of player *j* is the marginal contribution to the value when it enters any coalition
- Produce all possible joining orders of player coalitions
- Measure and average the difference in payout after player 1 enters the coalition



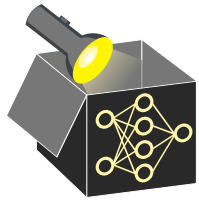
# SHAPLEY VALUES - ILLUSTRATION



- Shapley value of player *j* is the marginal contribution to the value when it enters any coalition
- Produce all possible joining orders of player coalitions
- Measure and average the difference in payout after player 2 enters the coalition



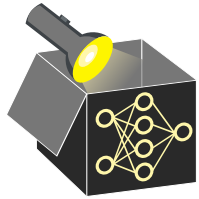
# SHAPLEY VALUES - ILLUSTRATION



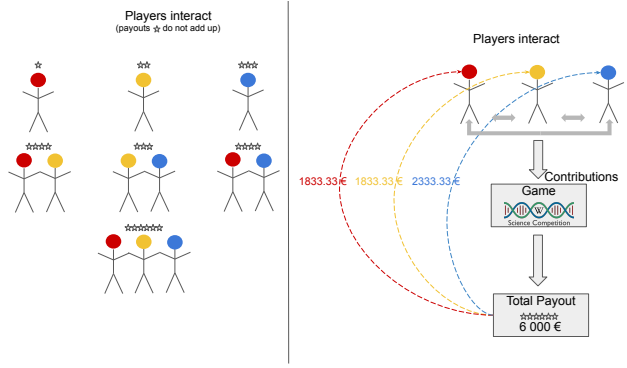
- Shapley value of player *j* is the marginal contribution to the value when it enters any coalition
- Produce all possible joining orders of player coalitions
- Measure and average the difference in payout after player 3 enters the coalition

| joining order | empty coalition | | player joins coalition | | player joins coalition | | player joins coalition | contribution of player 3 |
|---|---|---|---|---|---|---|---|---|
| 1, 2, 3 | S = ∅ 0 | 1 | S = {1} +1000 | 2 | S = {1, 2} +4000 | 3 | S = {1, 2, 3} +6000 | +2000 |
| 1, 3, 2 | S = ∅ 0 | 1 | S = {1} +1000 | 3 | S = {1, 3} +4000 | 2 | S = {1, 2, 3} +6000 | +3000 |
| 2, 1, 3 | S = ∅ 0 | 2 | S = {2} +2000 | 1 | S = {1, 2} +4000 | 3 | S = {1, 2, 3} +6000 | +2000 |
| 2, 3, 1 | S = ∅ 0 | 2 | S = {2} +2000 | 3 | S = {2, 3} +3000 | 1 | S = {1, 2, 3} +6000 | +1000 |
| 3, 1, 2 | S = ∅ 0 | 3 | S = {3} +3000 | 1 | S = {1, 3} +4000 | 2 | S = {1, 2, 3} +6000 | +3000 |
| 3, 2, 1 | S = ∅ 0 | 3 | S = {3} +3000 | 2 | S = {2, 3} +3000 | 1 | S = {1, 2, 3} +6000 | +3000 |
| | | | | | | | Shapley value of player 3 | +2333.33 |

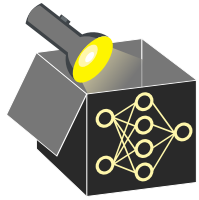
# SHAPLEY VALUES - ILLUSTRATION



- Shapley value of player *j* is the marginal contribution to the value when it enters any coalition
- Produce all possible joining orders of player coalitions



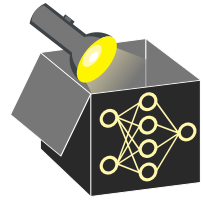
# AXIOMS OF FAIR PAYOUTS



Why is this a fair payout solution?
One possibility to define fair payouts are the following axioms for a given value function $v$:

- **Efficiency**: Player contributions add up to the total payout of the game: $\sum_{j=1}^{p} \phi_j = v(P)$

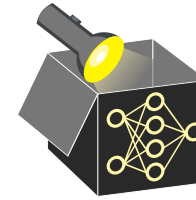
# AXIOMS OF FAIR PAYOUTS



Why is this a fair payout solution?
One possibility to define fair payouts are the following axioms for a given value function $v$:

- **Efficiency**: Player contributions add up to the total payout of the game: $\sum_{j=1}^{p} \phi_j = v(P)$
- **Symmetry**: Players $j, k \in P$ who contribute the same to any coalition get the same payout:
  If $v(S \cup \{j\}) = v(S \cup \{k\})$ for all $S \subseteq P \setminus \{j, k\}$, then $\phi_j = \phi_k$

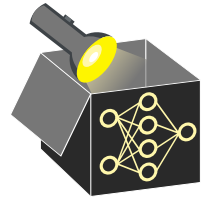
# AXIOMS OF FAIR PAYOUTS



Why is this a fair payout solution?
One possibility to define fair payouts are the following axioms for a given value function $v$:

- **Efficiency**: Player contributions add up to the total payout of the game:
  $\sum_{j=1}^{p} \phi_j = v(P)$
- **Symmetry**: Players $j, k \in P$ who contribute the same to any coalition get the same payout:
  If $v(S \cup \{j\}) = v(S \cup \{k\})$ for all $S \subseteq P \setminus \{j, k\}$, then $\phi_j = \phi_k$
- **Dummy/Null Player**: Payout is 0 for players who don't contribute to the value of any coalition:
  If $v(S \cup \{j\}) = v(S) \quad \forall \quad S \subseteq P \setminus \{j\}$, then $\phi_j = 0$

# AXIOMS OF FAIR PAYOUTS



Why is this a fair payout solution?
One possibility to define fair payouts are the following axioms for a given value function $v$:

- **Efficiency**: Player contributions add up to the total payout of the game: $\sum_{j=1}^{p} \phi_j = v(P)$
- **Symmetry**: Players $j, k \in P$ who contribute the same to any coalition get the same payout:
  If $v(S \cup \{j\}) = v(S \cup \{k\})$ for all $S \subseteq P \setminus \{j, k\}$, then $\phi_j = \phi_k$
- **Dummy/Null Player**: Payout is 0 for players who don't contribute to the value of any coalition:
  If $v(S \cup \{j\}) = v(S) \quad \forall \quad S \subseteq P \setminus \{j\}$, then $\phi_j = 0$
- **Additivity**: For a game $v$ with combined payouts $v(S) = v_1(S) + v_2(S)$, the payout is the sum of payouts: $\phi_{j,v} = \phi_{j,v_1} + \phi_{j,v_2}$