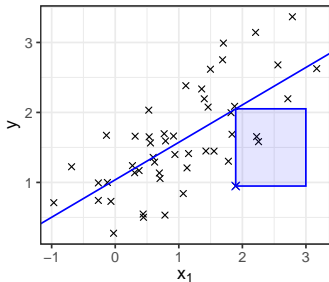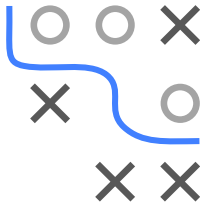# Introduction to Machine Learning

Supervised Regression
Linear Models with $L2$ Loss

Learning goals

- Grasp the overall concept of linear regression
- Understand how $L2$ loss optimization results in SSE-minimal model
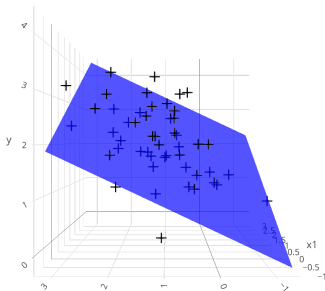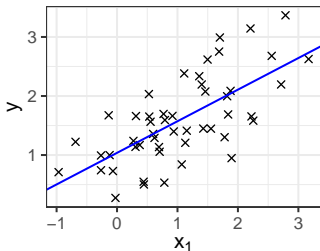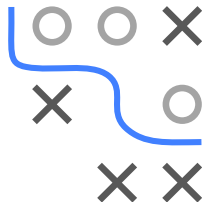- Understand this as a general template for ERM in ML

LINEAR REGRESSION

- Idea: predict $y \in \mathbb{R}$ as **linear** combination of features[1]:

$$\hat{y} = f(\mathbf{x}) = \boldsymbol{\theta}^\top \mathbf{x} = \theta_0 + \theta_1 x_1 + \cdots + \theta_p x_p$$

  $\rightsquigarrow$ find loss-optimal params to describe relation $y|\mathbf{x}$

- Hypothesis space: $\mathcal{H} = \{f(\mathbf{x}) = \boldsymbol{\theta}^\top \mathbf{x} \mid \boldsymbol{\theta} \in \mathbb{R}^{p+1}\}$


---

[1] Actually, special case of linear model, which is linear combo of *basis functions* of features $\rightsquigarrow$ Polynomial Regression Models

DESIGN MATRIX

- Mismatch: $\theta \in \mathbb{R}^{p+1}$ vs $\mathbf{x} \in \mathbb{R}^p$ due to intercept term
- Trick: pad feature vectors with leading 1, s.t.
    - $\mathbf{x} \mapsto \mathbf{x} = (1, x_1, \ldots, x_p)^\top$, and
    - $\theta^\top \mathbf{x} = \theta_0 \cdot 1 + \theta_1 x_1 + \cdots + \theta_p x_p$
- Collect all observations in **design matrix** $\mathbf{X} \in \mathbb{R}^{n \times (p+1)}$
  $\rightsquigarrow$ more compact: single param vector incl. intercept
- Resulting linear model:

$$\hat{\mathbf{y}} = \mathbf{X}\theta = \begin{pmatrix} 1 & x_1^{(1)} & \ldots & x_p^{(1)} \\ 1 & x_1^{(2)} & \ldots & x_p^{(2)} \\ \vdots & \vdots & & \vdots \\ 1 & x_1^{(n)} & \ldots & x_p^{(n)} \end{pmatrix} \begin{pmatrix} \theta_0 \\ \theta_1 \\ \vdots \\ \theta_p \end{pmatrix} = \begin{pmatrix} \theta_0 + \theta_1 x_1^{(1)} + \cdots + \theta_p x_p^{(1)} \\ \theta_0 + \theta_1 x_1^{(2)} + \cdots + \theta_p x_p^{(2)} \\ \vdots \\ \theta_0 + \theta_1 x_1^{(n)} + \cdots + \theta_p x_p^{(n)} \end{pmatrix}$$

- We will make use of this notation in other contexts
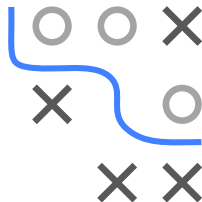
EFFECT INTERPRETATION

- Big plus of LM: immediately **interpretable** feature effects
- "Marginally increasing x_j by 1 unit increases y by θ_j units"
  - $\leadsto$ *ceteris paribus* assumption: $x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_p$ fixed
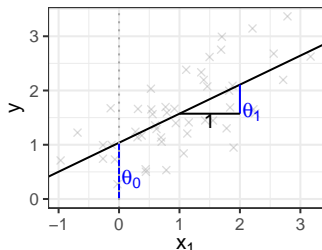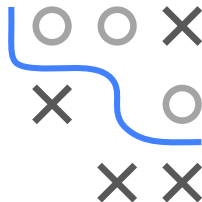

```
Call:
lm(formula = y ~ x_1, data = dt_univ)

Residuals:
     Min       1Q   Median       3Q      Max
-1.10346 -0.34727 -0.00766  0.31500  1.04284

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)  1.03727    0.11360   9.131 4.55e-12 ***
x_1          0.53521    0.08219   6.512 4.13e-08 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 0.5327 on 48 degrees of freedom
Multiple R-squared:  0.469,    Adjusted R-squared:  0.458
F-statistic: 42.4 on 1 and 48 DF,  p-value: 4.129e-08
```
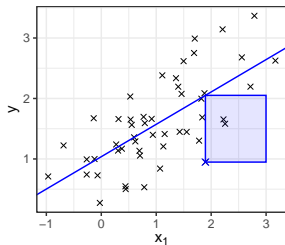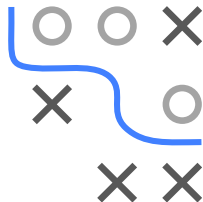
MODEL FIT

- How to determine LM fit? $\rightsquigarrow$ define risk & optimize
- Popular: $L2$ **loss** / **quadratic loss** / **squared error**

$$L\left(y, f(\mathbf{x})\right) = \left(y - f(\mathbf{x})\right)^2 \ \text{ or } \ L\left(y, f(\mathbf{x})\right) = 0.5 \cdot \left(y - f(\mathbf{x})\right)^2$$
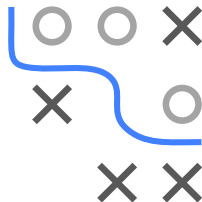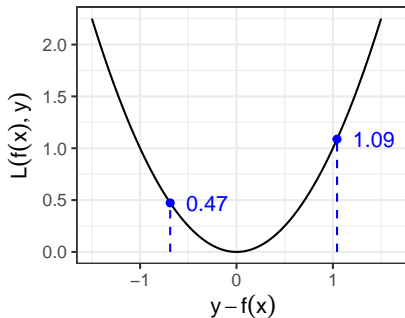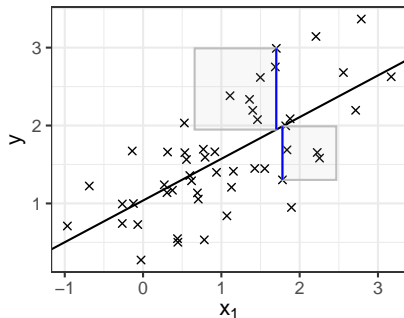

- Why penalize **residuals** $r = y - f(\mathbf{x})$ quadratically?
    - Easy to optimize (convex, differentiable)
    - Theoretically appealing (connection to classical stats LM)

LOSS PLOTS

We will often visualize loss effects like this:

- Data as $y \sim x_1$
- Prediction hypersurface
  ⤳ here: line
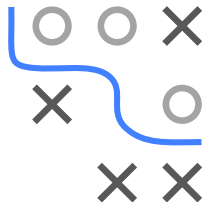- Residuals $r = y - f(\mathbf{x})$
  ⤳ squares to illustrate loss

- Loss as function of residuals
  ⤳ strength of penalty?
  ⤳ symmetric?
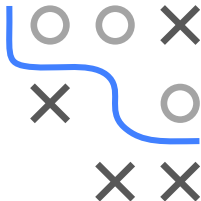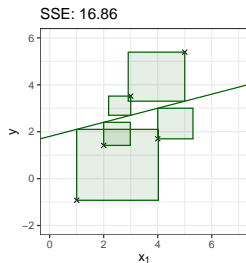- Highlighted: loss for
  residuals shown on LHS

OPTIMIZATION

- Resulting risk equivalent to **sum of squared errors (SSE)**:

$$\mathcal{R}_{\text{emp}}(\boldsymbol{\theta}) = \sum_{i=1}^{n} \left(y^{(i)} - \boldsymbol{\theta}^{\top} \mathbf{x}^{(i)} \right)^2$$

- Consider example with $n = 5 \rightsquigarrow$ different models with varying SSE

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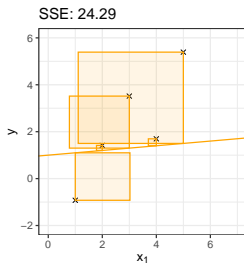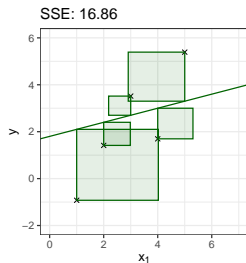- Consider example with $n = 5 \rightsquigarrow$ different models with varying SSE

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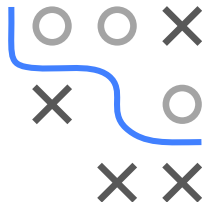- Consider example with $n = 5 \rightsquigarrow$ different models with varying SSE
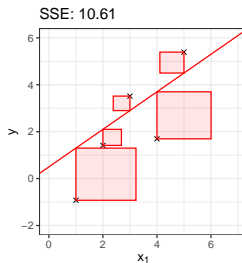
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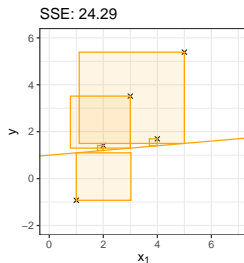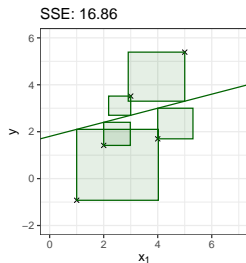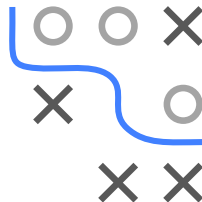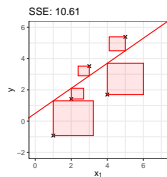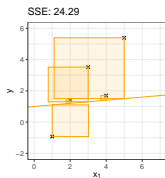- Consider example with $n = 5 \rightsquigarrow$ different models with varying SSE

OPTIMIZATION

Intercept θ_0	Slope θ_1	SSE
1.80	0.30	16.86
1.00	0.10	24.29
0.50	0.80	10.61

OPTIMIZATION
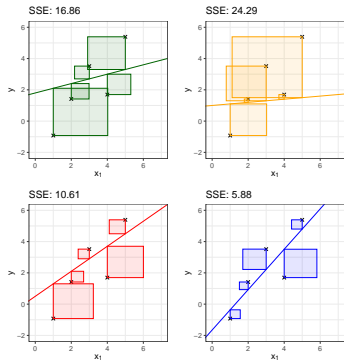
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-1.65	1.29	5.88
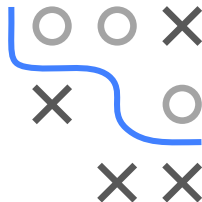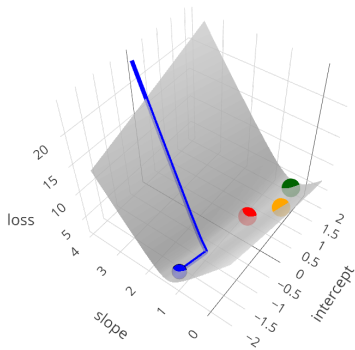
Instead of guessing, of course, use **optimization**!

ANALYTICAL OPTIMIZATION

- Special property of LM with *L2* loss: **analytical solution** available

$$\hat{\boldsymbol{\theta}} \in \underset{\boldsymbol{\theta}}{\arg\min}\, \mathcal{R}_{\mathsf{emp}}(\boldsymbol{\theta}) = \underset{\boldsymbol{\theta}}{\arg\min} \sum_{i=1}^{n} \left(y^{(i)} - \boldsymbol{\theta}^{\top}\mathbf{x}^{(i)} \right)^2$$
$$= \underset{\boldsymbol{\theta}}{\arg\min}\, \|\mathbf{y} - \mathbf{X}\boldsymbol{\theta}\|_2^2$$

- Find via **normal equations**

$$\frac{\partial \mathcal{R}_{\mathsf{emp}}(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} = 0$$

- Solution: **ordinary-least-squares (OLS)** estimator

$$\hat{\boldsymbol{\theta}} = (\mathbf{X}^{\top}\mathbf{X})^{-1}\mathbf{X}^{\top}\mathbf{y}$$
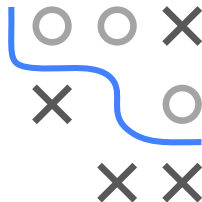
STATISTICAL PROPERTIES

- LM with *L2* loss intimately related to classical stats LM
- Assumptions
    - $\mathbf{x}^{(i)}$ **iid** for $i \in \{1, \ldots, n\}$
    - **Homoskedastic** (equivariant) **Gaussian** errors

    $$\mathbf{y} = \mathbf{X}\boldsymbol{\theta} + \boldsymbol{\epsilon}, \ \boldsymbol{\epsilon} \sim \mathcal{N}(0, \sigma^2\boldsymbol{I})$$

    $\rightsquigarrow$ y_i conditionally independent & normal: $\mathbf{y}|\mathbf{X} \sim \mathcal{N}(\mathbf{X}\boldsymbol{\theta}, \sigma^2\boldsymbol{I})$
    - Uncorrelated features
        $\rightsquigarrow$ multicollinearity destabilizes effect estimation
- If assumptions hold: statistical **inference** applicable
    - Hypothesis tests on significance of effects, incl. *p*-values
    - Confidence & prediction intervals via student-*t* distribution
    - Goodness-of-fit measure $R^2 = 1 - \text{SSE} \ / \ \underbrace{\text{SST}}_{\sum\limits_{i=1}^{n}(y^{(i)} - \bar{y})^2}$

    $\rightsquigarrow$ SSE = part of data variance *not* explained by model