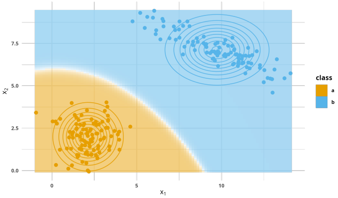
Introduction to Machine Learning

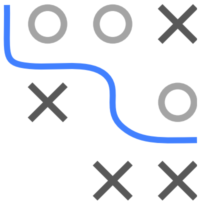
Classification
Naive Bayes



Learning goals

- Construction principle of NB
- Conditional independence assumption
- Numerical and categorical features
- Similarity to QDA, quadratic decision boundaries
- Laplace smoothing

NAIVE BAYES CLASSIFIER



Generative multiclass technique. Remember: We use Bayes' theorem and only need $p(\mathbf{x}|y = k)$ to compute the posterior as:

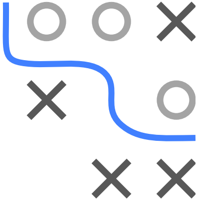
$$\pi_k(\mathbf{x}) \approx \mathbb{P}(y = k \mid \mathbf{x}) = \frac{\mathbb{P}(\mathbf{x}|y = k)\mathbb{P}(y = k)}{\mathbb{P}(\mathbf{x})} = \frac{p(\mathbf{x}|y = k)\pi_k}{\sum\limits_{j=1}^{g} p(\mathbf{x}|y = j)\pi_j}$$

NB is based on a simple **conditional independence assumption**: the features are conditionally independent given class y.

$$p(\mathbf{x}|y = k) = p((x_1, x_2, ..., x_p)|y = k) = \prod_{j=1}^{p} p(x_j|y = k).$$

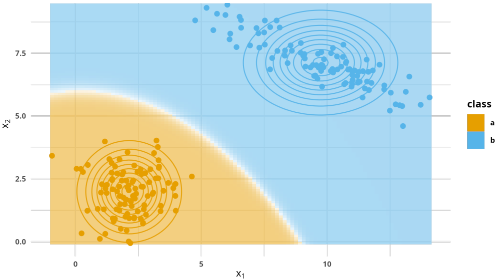
So we only need to specify and estimate the distributions $p(x_j|y = k)$, which is considerably simpler as these are univariate.

NUMERICAL FEATURES



Use univariate Gaussians for $p(x_j|y=k)$, and estimate $(\mu_{kj}, \sigma^2_{kj})$.

Because of $p(\mathbf{x}|y=k) = \prod_{j=1}^{p} p(x_j|y=k)$, joint conditional density is Gaussian with diagonal, non-isotropic covariances, and different across classes, so **QDA with diagonal covariances**.



Note: In the above plot the data violates the NB assumption.

NB: CATEGORICAL FEATURES

We use a categorical distribution for $p(x_j|y=k)$ and estimate the probabilities p_{kjm} that, in class k, our j-th feature has value m, $x_j = m$, simply by counting frequencies.

$$p(x_j|y=k) = \prod_m p_{kjm}^{[x_j=m]}$$

Because of the simple conditional independence structure, it is also very easy to deal with mixed numerical / categorical feature spaces.

ID	Class	Sex	Survived the Titanic
1	2nd	male	no
2	1st	male	yes
3	3rd	female	yes
4	1st	female	yes
5	2nd	female	yes
6	3rd	female	no

$$p(x_{\text{sex}}|y=\text{yes}) = p_{yes,sex,female}^{[x_{\text{sex}}=\text{female}]} \cdot p_{yes,sex,male}^{[x_{\text{sex}}=\text{male}]} = \frac{3}{4}^{[x_{\text{sex}}=\text{female}]} \cdot \frac{1}{4}^{[x_{\text{sex}}=\text{male}]}$$

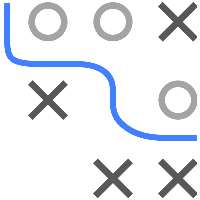
LAPLACE SMOOTHING

If a given class and feature value never occur together in the training data, then the frequency-based probability estimate will be zero, e.g.:
$p_{\text{no, class, 1st}}^{[x_{class}=\text{1st}]} = 0$ (everyone from 1st class survived in the previous table)

This is problematic because it will wipe out all information in the other probabilities when they are multiplied!

$$\pi_{no}(\text{class = 1st, sex = male}) = \frac{\hat{p}(x_{class}|y = no) \cdot \hat{p}(x_{sex}|y = no) \cdot \hat{\pi}_{no}}{\sum_{j=1}^{g} \hat{p}(\text{class = 1st, sex = male}|y = j)\hat{\pi}_j} = 0$$

LAPLACE SMOOTHING



A simple numerical correction is to set these zero probabilities to a small value to regularize against this case.

- Add constant $\alpha > 0$ (e.g., $\alpha = 1$).
- For a categorical feature x_j with M_j possible values:

$$p_{kjm}^{[x_j=m]} = \frac{n_{kjm} + \alpha}{n_k + \alpha M_j} \quad \left(\text{instead of } p_{kjm}^{[x_j=m]} = \frac{n_{kjm}}{n_k}\right)$$

  where:

  - n_{kjm}: count of $x_j = m$ in class k,
  - n_k: total counts in class k,
  - M_j: number of possible distinct values of x_j.

This ensures that our posterior probabilities are non-zero due to such effects, preserving the influence of all features in the model.

NAIVE BAYES: APPLICATION AS SPAM FILTER

- In the late 90s, NB became popular for e-mail spam detection
- Word counts were used as features to detect spam mails
- Independence assumption implies: occurrence of two words in mail is not correlated, this is often wrong;
  "viagra" more likely to occur in context with "buy"...
- In practice: often still good performance

Benchmarking QDA, NB and LDA on `spam`: