FTL FOR OQO PROBLEMS

- One popular instantiation of the online learning problem is the problem of *online quadratic optimization* (OQO).
- In its most general form, the loss function is thereby defined as

$$(a_t, z_t) = \frac{1}{2} \, ||a_t - z_t||_2^2 \, ,$$

where $\mathcal{A}, \mathcal{Z} \subset \mathbb{R}^d$.

- **Proposition:** Using FTL on any online quadratic optimization problem with $\mathcal{A} = \mathbb{R}^d$ and $V = \sup_{z \in \mathcal{Z}} ||z||_2$, leads to a regret of

$$R_T^{\mathtt{FTL}} \leq 4V^2 \, (\log(T) + 1).$$

FTL FOR OQO PROBLEMS: ANALYSIS

- **Proof:**
  - In the following, we denote $a_1^{\mathtt{FTL}}, a_2^{\mathtt{FTL}}, \ldots$ simply by $a_1, a_2, \ldots$

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Reminder (Useful Lemma):

$$R_T^{\mathtt{FTL}} \leq \sum_{t=1}^{T} (a_t^{\mathtt{FTL}}, z_t) - \sum_{t=1}^{T} (a_{t+1}^{\mathtt{FTL}}, z_t)$$

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  - Using this lemma, we just have to show that

$$\sum_{t=1}^{T} ((a_t, z_t) - (a_{t+1}, z_t)) \leq 4L^2 \cdot (\log(T) + 1)\,. \qquad (1)$$

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  - So, we will prove (1). For this purpose, we compute the explicit form of the actions of FTL for this type of online learning problem.

FTL FOR OQO PROBLEMS: ANALYSIS

- Claim: It holds that $a_t = \frac{1}{t-1} \cdot \sum_{s=1}^{t-1} z_s$, if $(a, z) = \frac{1}{2} ||a - z||_2^2$.

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  - Recall that

$$a_t^{\mathrm{FTL}} = \arg\min_{a \in \mathcal{A}} \sum_{s=1}^{t-1} (a, z_s) = \arg\min_{a \in \mathcal{A}} \sum_{s=1}^{t-1} \frac{1}{2} \left|\left|a - z_s\right|\right|_2^2 .$$

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  - So, we have to find the minimizer of the function

$$f(a) := \sum_{s=1}^{t-1} \frac{1}{2} \left|\left|a - z_s\right|\right|_2^2 = \sum_{s=1}^{t-1} \frac{1}{2} (a - z_s)^\top (a - z_s).$$

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  - Compute $\nabla f(a) = \sum_{s=1}^{t-1} a - z_s = (t-1)a - \sum_{s=1}^{t-1} z_s$, which we set to zero and solve with respect to a to obtain the claim.

    (f is convex, so that this leads indeed to a minimizer.)

FTL FOR OQO PROBLEMS: ANALYSIS

- Hence, a_t is the empirical average of $z_1, \ldots, z_{t-1}$ and we can provide the following incremental update formula for its computation

$$
\begin{aligned}
a_{t+1} = \tfrac{1}{t} \cdot \sum_{s=1}^{t} z_s &= \tfrac{1}{t}\left(z_t + \sum_{s=1}^{t-1} z_s\right) \\
&= \tfrac{1}{t}(z_t + (t-1)a_t) = \tfrac{1}{t} z_t + \left(1 - \tfrac{1}{t}\right) a_t .
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- From the last display we derive that

$$a_{t+1} - z_t = \left(1 - \tfrac{1}{t}\right) \cdot a_t + \tfrac{1}{t} z_t - z_t = \left(1 - \tfrac{1}{t}\right) \cdot (a_t - z_t).$$

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- Claim:

$$(a_t, z_t) - (a_{t+1}, z_t) \leq \frac{1}{t} \cdot ||a_t - z_t||_2^2 . \tag{2}$$

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Reminder: $a_{t+1} - z_t = \left(1 - \frac{1}{t}\right) \cdot (a_t - z_t).$

- Indeed, this can be seen as follows

$$\begin{aligned}
(a_t, z_t) - (a_{t+1}, z_t) &= \tfrac{1}{2} \left|\left|a_t - z_t\right|\right|_2^2 - \frac{1}{2} \left|\left|a_{t+1} - z_t\right|\right|_2^2 \\
&= \tfrac{1}{2} \left(\left|\left|a_t - z_t\right|\right|_2^2 - \left|\left|a_{t+1} - z_t\right|\right|_2^2 \right) \\
&= \tfrac{1}{2} \left(\left|\left|a_t - z_t\right|\right|_2^2 - \left|\left|\left(1 - \tfrac{1}{t}\right) \cdot (a_t - z_t)\right|\right|_2^2 \right).
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\end{aligned}
$$

- And from this,

$$
\begin{aligned}
(a_t, z_t) - (a_{t+1}, z_t) &= \tfrac{1}{2}\left(||a_t - z_t||_2^2 - \left(1 - \tfrac{1}{t}\right)^2 \cdot ||a_t - z_t||_2^2\right) \\
&= \tfrac{1}{2}\left(1 - \left(1 - \tfrac{1}{t}\right)^2\right) \cdot ||a_t - z_t||_2^2 \\
&= \left(\tfrac{1}{t} - \tfrac{1}{2t^2}\right) \cdot ||a_t - z_t||_2^2 \\
&\leq \tfrac{1}{t} \cdot ||a_t - z_t||_2^2 .
\end{aligned}
$$

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$$\textbf{Reminder:} \quad (a_t, z_t) - (a_{t+1}, z_t) \leq \frac{1}{t} \cdot ||a_t - z_t||_2^2 \, . \qquad (2)$$

- Since by assumption $L = \sup_{z \in \mathcal{Z}} ||z||_2$ and a_t is the empirical average of $z_1, \ldots, z_{t-1}$, we have that $||a_t||_2 \leq L$.

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- Now the triangle inequality states that for any two vectors $x, y \in \mathbb{R}^d$ it holds that

  $$||x + y||_2 \leq ||x||_2 + ||y||_2 \,,$$

  so that

  $$||a_t - z_t||_2 \leq ||a_t||_2 + ||z_t||_2 \leq 2L. \tag{3}$$

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so that
$$||a_t - z_t||_2 \leq ||a_t||_2 + ||z_t||_2 \leq 2L. \qquad (3)$$
- Summing over all t in (2) and using (3) we arrive at

$$\sum_{t=1}^{T} \left((a_t, z_t) - (a_{t+1}, z_t) \right) \leq \sum_{t=1}^{T} \left(\frac{1}{t} \cdot ||a_t - z_t||_2^2 \right) \leq \sum_{t=1}^{T} \frac{1}{t} \cdot (2L)^2$$
$$= 4L^2 \cdot \sum_{t=1}^{T} \frac{1}{t}.$$

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- Now, it holds that $\sum_{t=1}^{T} \frac{1}{t} \leq \log(T) + 1$, so that we obtain

$$\sum_{t=1}^{T} ((a_t, z_t) - (a_{t+1}, z_t)) \leq 4L^2 \cdot \sum_{t=1}^{T} \frac{1}{t} \leq 4L^2 \cdot (\log(T) + 1)\,,$$

which is what we wanted to prove. $\square$