FTL FOR ONLINE LINEAR OPTIMIZATION

- Another popular instantiation of the online learning problem is the online linear optimization problem, which is characterized by a linear loss function $(a, z) = a^\top z$.

FTL FOR ONLINE LINEAR OPTIMIZATION

- Another popular instantiation of the online learning problem is the online linear optimization problem, which is characterized by a linear loss function $(a, z) = a^\top z$.

- Let $\mathcal{A} = [-1, 1]$ and suppose that $z_t = \begin{cases} -\frac{1}{2}, & t = 1, \\ 1, & t \text{ is even}, \\ -1, & t \text{ is odd}. \end{cases}$

FTL FOR ONLINE LINEAR OPTIMIZATION

- Another popular instantiation of the online learning problem is the online linear optimization problem, which is characterized by a linear loss function $(a, z) = a^\top z$.

- Let $\mathcal{A} = [-1, 1]$ and suppose that $z_t = \begin{cases} -\frac{1}{2}, & t = 1, \\ 1, & t \text{ is even}, \\ -1, & t \text{ is odd}. \end{cases}$

- No matter how we choose the first action a_1^{FTL}, it will hold that FTL has a cumulative loss greater than (or equal) $T - 3/2$, while the best action in hindsight has a cumulative loss of $-1/2$.

- Thus, FTL's cumulative regret is at least $T - 1$, which is linearly growing in T.

FTL FOR ONLINE LINEAR OPTIMIZATION

- Indeed, note that

$$a_{t+1}^{\texttt{FTL}} = \arg\min_{a \in \mathcal{A}} \sum_{s=1}^{t} (a, z_s) = \arg\min_{a \in [-1,1]} a \sum_{s=1}^{t} z_s$$

$$= \begin{cases} -1, & \text{if } \sum_{s=1}^{t} z_s > 0, \\ 1, & \text{if } \sum_{s=1}^{t} z_s < 0, \\ \text{arbitrary}, & \text{if } \sum_{s=1}^{t} z_s = 0. \end{cases}$$

FTL FOR ONLINE LINEAR OPTIMIZATION

- Indeed, note that

$$a_{t+1}^{\mathtt{FTL}} = \arg\min_{a \in \mathcal{A}} \sum_{s=1}^{t} (a, z_s) = \arg\min_{a \in [-1,1]} a \sum_{s=1}^{t} z_s$$

$$= \begin{cases} -1, & \text{if } \sum_{s=1}^{t} z_s > 0, \\ 1, & \text{if } \sum_{s=1}^{t} z_s < 0, \\ \text{arbitrary}, & \text{if } \sum_{s=1}^{t} z_s = 0. \end{cases}$$

t	$a_t^{\mathtt{FTL}}$	z_t	$(a_t^{\mathtt{FTL}}, z_t)$	$\sum_{s=1}^{t} (a_s^{\mathtt{FTL}}, z_s)$	$\sum_{s=1}^{t} z_s$
1	1	$-1/2$	$-1/2$	$-1/2$	$-1/2$

FTL FOR ONLINE LINEAR OPTIMIZATION

- Indeed, note that

$$a_{t+1}^{\texttt{FTL}} = \arg\min_{a \in \mathcal{A}} \sum_{s=1}^{t} (a, z_s) = \arg\min_{a \in [-1,1]} a \sum_{s=1}^{t} z_s$$

$$= \begin{cases} -1, & \text{if } \sum_{s=1}^{t} z_s > 0, \\ 1, & \text{if } \sum_{s=1}^{t} z_s < 0, \\ \text{arbitrary}, & \text{if } \sum_{s=1}^{t} z_s = 0. \end{cases}$$

t	$a_t^{\texttt{FTL}}$	z_t	$(a_t^{\texttt{FTL}}, z_t)$	$\sum_{s=1}^{t} (a_s^{\texttt{FTL}}, z_s)$	$\sum_{s=1}^{t} z_s$
1	1	$-1/2$	$-1/2$	$-1/2$	$-1/2$
2	1	1	1	$1 - 1/2$	1/2

FTL FOR ONLINE LINEAR OPTIMIZATION

- Indeed, note that

$$a_{t+1}^{\mathtt{FTL}} = \arg\min_{a \in \mathcal{A}} \sum_{s=1}^{t} (a, z_s) = \arg\min_{a \in [-1,1]} a \sum_{s=1}^{t} z_s$$

$$= \begin{cases} -1, & \text{if } \sum_{s=1}^{t} z_s > 0, \\ 1, & \text{if } \sum_{s=1}^{t} z_s < 0, \\ \text{arbitrary}, & \text{if } \sum_{s=1}^{t} z_s = 0. \end{cases}$$

t	$a_t^{\mathtt{FTL}}$	z_t	$(a_t^{\mathtt{FTL}}, z_t)$	$\sum_{s=1}^{t} (a_s^{\mathtt{FTL}}, z_s)$	$\sum_{s=1}^{t} z_s$
1	1	$-1/2$	$-1/2$	$-1/2$	$-1/2$
2	1	1	1	$1 - 1/2$	1/2
3	-1	-1	1	$2 - 1/2$	$-1/2$

FTL FOR ONLINE LINEAR OPTIMIZATION

- Indeed, note that

$$a_{t+1}^{\texttt{FTL}} = \arg\min_{a \in \mathcal{A}} \sum_{s=1}^{t} (a, z_s) = \arg\min_{a \in [-1,1]} a \sum_{s=1}^{t} z_s$$

$$= \begin{cases} -1, & \text{if } \sum_{s=1}^{t} z_s > 0, \\ 1, & \text{if } \sum_{s=1}^{t} z_s < 0, \\ \text{arbitrary}, & \text{if } \sum_{s=1}^{t} z_s = 0. \end{cases}$$

t	$a_t^{\texttt{FTL}}$	z_t	$(a_t^{\texttt{FTL}}, z_t)$	$\sum_{s=1}^{t} (a_s^{\texttt{FTL}}, z_s)$	$\sum_{s=1}^{t} z_s$
1	1	$-1/2$	$-1/2$	$-1/2$	$-1/2$
2	1	1	1	$1 -1/2$	$1/2$
3	-1	-1	1	$2 -1/2$	$-1/2$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
T	$(-1)^T$	$(-1)^T$	1	$T - 1 - 1/2$	$(-1/2)^T$

FTL FOR ONLINE LINEAR OPTIMIZATION

- Indeed, note that

$$a_{t+1}^{\texttt{FTL}} = \arg\min_{a \in \mathcal{A}} \sum_{s=1}^{t} (a, z_s) = \arg\min_{a \in [-1,1]} a \sum_{s=1}^{t} z_s$$

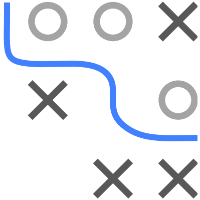
$$= \begin{cases} -1, & \text{if } \sum_{s=1}^{t} z_s > 0, \\ 1, & \text{if } \sum_{s=1}^{t} z_s < 0, \\ \text{arbitrary}, & \text{if } \sum_{s=1}^{t} z_s = 0. \end{cases}$$

t	$a_t^{\texttt{FTL}}$	z_t	$(a_t^{\texttt{FTL}}, z_t)$	$\sum_{s=1}^{t} (a_s^{\texttt{FTL}}, z_s)$	$\sum_{s=1}^{t} z_s$
1	1	$-1/2$	$-1/2$	$-1/2$	$-1/2$
2	1	1	1	$1 - 1/2$	$1/2$
3	-1	-1	1	$2 - 1/2$	$-1/2$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
T	$(-1)^T$	$(-1)^T$	1	$T - 1 - 1/2$	$(-1/2)^T$

- The best action has cumulative loss

$$\inf_{a \in \mathcal{A}} \sum_{s=1}^{T} (a, z_s) = \inf_{a \in [-1,1]} a \underbrace{\sum_{s=1}^{T} z_s}_{=(-1/2)^T} = -1/2.$$

FTL FOR ONLINE LINEAR OPTIMIZATION



- Thus, we see: FTL can fail for online linear optimization problems, although it is well suited for online quadratic optimization problems!
- The reason is that the action selection of FTL is not stable enough (caused by the loss function), which is fine for the latter problem, but problematic for the former.
- One has to note that the online linear optimization problem example above, where FTL fails, is in fact an adversarial learning setting: The environmental data is generated in such a way that the FTL learner is fooled in each time step.