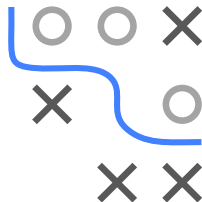


FTL FOR ONLINE LINEAR OPTIMIZATION

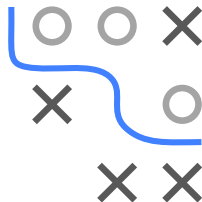
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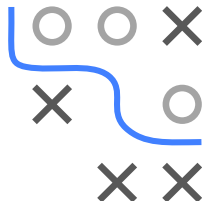
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- No matter how we choose the first action a_1^{FTL} , it will hold that FTL has a cumulative loss greater than (or equal) $T - 3/2$, while the best action in hindsight has a cumulative loss of $-1/2$.

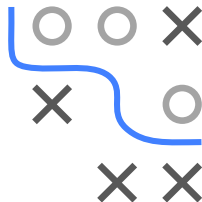
- Thus, FTL's cumulative regret is at least $T - 1$, which is linearly growing in T .



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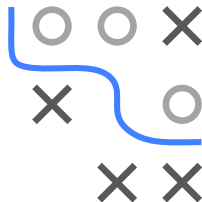


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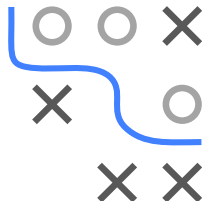


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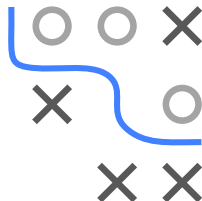


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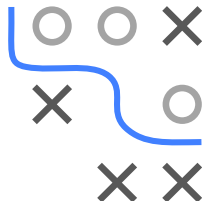


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\vdots	\vdots	\vdots	\vdots	\vdots	\vdots
T	$(-1)^T$	$(-1)^T$	1	$T - 1 - 1/2$	$(-1/2)^T$



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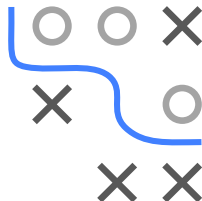
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- The best action has cumulative loss

$$\inf_{a \in \mathcal{A}} \sum_{s=1}^T (a, z_s) = \inf_{a \in [-1, 1]} a \underbrace{\sum_{s=1}^T z_s}_{=(-1/2)^T} = -1/2.$$



FTL FOR ONLINE LINEAR OPTIMIZATION

- Thus, we see: FTL can fail for **online linear optimization problems**, although it is well suited for **online quadratic optimization problems**!
- The reason is that the action selection of FTL is not stable enough (caused by the loss function), which is fine for **the latter problem**, but problematic for **the former**.
- One has to note that the online linear optimization problem example above, where FTL fails, is in fact an adversarial learning setting: The environmental data is generated in such a way that the FTL learner is fooled in each time step.

