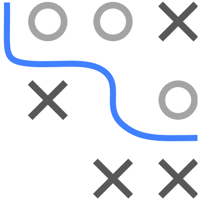
ONLINE CONVEX OPTIMIZATION



- One of the most relevant instantiations of the online learning problem is the problem of *online convex optimization* (OCO), which is characterized by a loss function : $\mathcal{A} \times \mathcal{Z} \to \mathbb{R}$, which is convex w.r.t. the action, i.e., $a \mapsto (a, z)$ is convex for any $z \in \mathcal{Z}$.

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- Note that both OLO and OQO belong to the class of online convex optimization problems:
    - *Online linear optimization (OLO) with convex action spaces:* $(a, z) = a^\top z$ is a convex function in $a \in \mathcal{A}$, provided $\mathcal{A}$ is convex.
    - *Online quadratic optimization (OQO) with convex action spaces:* $(a, z) = \frac{1}{2} ||a - z||_2^2$ is a convex function in $a \in \mathcal{A}$, provided $\mathcal{A}$ is convex.

ONLINE GRADIENT DESCENT: MOTIVATION

- We have seen that the FTRL algorithm with the $_2$ norm regularization $\psi(a) = \frac{1}{2\eta}\,||a||_2^2$ achieves satisfactory results for online linear optimization (OLO) problems, that is, if $(a, z) = L^{\mathtt{lin}}(a, z) := a^\top z$, then we have
  - *Fast updates* — If $\mathcal{A} = \mathbb{R}^d$, then

$$a_{t+1}^{\mathtt{FTRL}} = a_t^{\mathtt{FTRL}} - \eta\, z_t, \qquad t = 1, \ldots, T;$$

  - *Regret bounds* — By an appropriate choice of η and some (mild) assumptions on $\mathcal{A}$ and $\mathcal{Z}$, we have

$$R_T^{\mathtt{FTRL}} = o(T).$$

ONLINE GRADIENT DESCENT: MOTIVATION

Apparently, the nice form of the loss function $L^{\mathtt{lin}}$ is responsible for the appealing properties of FTRL in this case. Indeed, since $\nabla_a L^{\mathtt{lin}}(a, z) = z$ note that the update rule can be written as

$$a_{t+1}^{\mathtt{FTRL}} = a_t^{\mathtt{FTRL}} - \eta\, z_t = a_t^{\mathtt{FTRL}} - \eta\, \nabla_a L^{\mathtt{lin}}(a_t^{\mathtt{FTRL}}, z_t).$$

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Interpretation: In each time step $t + 1$, we are following the direction with the steepest decrease of the loss (represented by $-\nabla L^{\mathtt{lin}}(a_t^{\mathtt{FTRL}}, z_t)$) from the current "position" $a_t^{\mathtt{FTRL}}$ with the step size η

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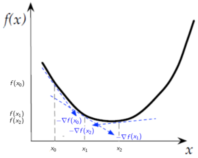

$\Rightarrow$ Gradient Descent.

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- **Question:** How to transfer this idea of the Gradient Descent for the update formula to other loss functions, while still preserving the regret bounds?

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- **Solution (for convex losses):** Recall the equivalent characterization of convexity of differentiable convex functions:

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\begin{aligned}
f : S \to \mathbb{R} \text{ is convex } &\Leftrightarrow f(y) \geq f(x) + (y - x)^\top \nabla f(x) \text{ for any } x, y \in S \\
&\Leftrightarrow f(x) - f(y) \leq (x - y)^\top \nabla f(x) \text{ for any } x, y \in S.
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- This means if we are dealing with a loss function $: \mathcal{A} \times \mathcal{Z} \to \mathbb{R}$, which is convex and differentiable in its first argument ($\mathcal{A}$ has also to be convex), then

$$(a, z) - (\tilde{a}, z) \leq (a - \tilde{a})^\top \nabla_a(a, z), \quad \forall a, \tilde{a} \in \mathcal{A}, z \in \mathcal{Z}.$$

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- Let $z_1, \ldots, z_T$ arbitrary environmental data and $a_1, \ldots, a_T$ be some arbitrary action sequence. Substitute $\tilde{z}_t := \nabla_a(a_t, z_t)$ and note that

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\begin{aligned}
{\color{blue}R_T(\tilde{a})} &= \sum_{t=1}^{T} (a_t, z_t) - (\tilde{a}, z_t) \leq \sum_{t=1}^{T} (a_t - \tilde{a})^\top \nabla_a(a_t, z_t) \\
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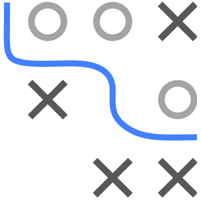

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Conclusion: The regret of a learner with respect to a differentiable and convex loss function is bounded by the regret corresponding to an online linear optimization problem with environmental data $\tilde{z}_t = \nabla_a (a_t, z_t)$.

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- **Reminder:** $\ell(a,z) - \ell(\tilde{a},z) \leq (a-\tilde{a})^\top \nabla_a \ell(a,z), \quad \forall a,\tilde{a} \in \mathcal{A}, z \in \mathcal{Z}.$
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- **We know:** Online linear optimization problems can be tackled by means of the FTRL algorithm!

$\rightsquigarrow$ Incorporate the substitution $\tilde{z}_t = \nabla_a(a_t, z_t)$ into the update formula of FTRL with squared L2-norm regularization.

ONLINE GRADIENT DESCENT: DEFINITION AND PROPERTIES

- The corresponding algorithm which chooses its action according to these considerations is called the *Online Gradient Descent* (OGD) algorithm with step size $\eta > 0$. It holds in particular,

$$a_{t+1}^{\mathtt{OGD}} = a_t^{\mathtt{OGD}} - \eta \nabla_a (a_t^{\mathtt{OGD}}, z_t), \quad t = 1, \ldots T. \tag{1}$$

(Technical side note: For this update formula we assume that $\mathcal{A} = \mathbb{R}^d$. Moreover, the first action $a_1^{\mathtt{OGD}}$ is arbitrary.)

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- We have the following connection between FTRL and OGD:
  - Let $\tilde{z}_t^{\mathtt{OGD}} := \nabla_a (a_t^{\mathtt{OGD}}, z_t)$ for any $t = 1, \dots, T$.
  - The update formula for FTRL with $_2$ norm regularization for the linear loss $L^{\mathtt{lin}}$ and the environmental data $\tilde{z}_t^{\mathtt{OGD}}$ is

$$a_{t+1}^{\mathtt{FTRL}} = a_t^{\mathtt{FTRL}} - \eta \tilde{z}_t^{\mathtt{OGD}} = a_t^{\mathtt{FTRL}} - \eta \nabla_a (a_t^{\mathtt{OGD}}, z_t).$$

  - If we have that $a_1^{\mathtt{FTRL}} = a_1^{\mathtt{OGD}}$, then it iteratively follows that $a_{t+1}^{\mathtt{FTRL}} = a_{t+1}^{\mathtt{OGD}}$ for any $t = 1, \dots, T$ in this case.

ONLINE GRADIENT DESCENT: DEFINITION AND PROPERTIES

- With the deliberations above we can infer that

$$
\begin{aligned}
R^{\texttt{OGD}}_{T,}(\tilde{a} \mid (z_t)_t) &= \sum\nolimits_{t=1}^{T} (a_t^{\texttt{OGD}}, z_t) - (\tilde{a}, z_t) \\
&\leq \sum\nolimits_{t=1}^{T} L^{\texttt{lin}}(a_t^{\texttt{OGD}}, \tilde{z}_t^{\texttt{OGD}}) - L^{\texttt{lin}}(\tilde{a}, \tilde{z}_t^{\texttt{OGD}}) \\
&\overset{(\text{if } a_1^{\texttt{OGD}} = a_1^{\texttt{FTRL}})}{=} \sum\nolimits_{t=1}^{T} L^{\texttt{lin}}(a_t^{\texttt{FTRL}}, \tilde{z}_t^{\texttt{OGD}}) - L^{\texttt{lin}}(\tilde{a}, \tilde{z}_t^{\texttt{OGD}}) \\
&= R^{\texttt{FTRL}}_{T,L^{\texttt{lin}}}(\tilde{a} \mid (\tilde{z}_t^{\texttt{OGD}})_t),
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where we write in the subscripts of the regret the corresponding loss function and also include the corresponding environmental data as a second argument in order to emphasize the connections.

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  where we write in the subscripts of the regret the corresponding loss function and also include the corresponding environmental data as a second argument in order to emphasize the connections.

- *Interpretation:* The regret of the FTRL algorithm (with $_2$ norm regularization) for the online linear optimization problem (characterized by the linear loss $L^{\mathtt{lin}}$) with environmental data $\tilde{z}_t^{\mathtt{OGD}}$ is an upper bound for the OGD algorithm for the online convex problem (characterized by a differentiable convex loss) with the original environmental data z_t.

ONLINE GRADIENT DESCENT: REGRET

- Due to this connection we immediately obtain a similar decomposition of the regret upper bound into a bias term and a variance term as for the FTRL algorithm for OLO problems.
- **Corollary.** Using the OGD algorithm on any online convex optimization problem (with differentiable loss function) leads to a regret of OGD with respect to any action $\tilde{a} \in \mathcal{A}$ of

$$\begin{aligned} R_T^{\texttt{OGD}}(\tilde{a}) &\leq \frac{1}{2\eta} \|\tilde{a}\|_2^2 + \eta \sum_{t=1}^{T} \left\|\tilde{z}_t^{\texttt{OGD}}\right\|_2^2 \\ &= \frac{1}{2\eta} \|\tilde{a}\|_2^2 + \eta \sum_{t=1}^{T} \left\|\nabla_a(a_t^{\texttt{OGD}}, z_t)\right\|_2^2 . \end{aligned}$$

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- Note that the step size $\eta > 0$ of OGD has the same role as the regularization magnitude of FTRL: It should balance the trade-off between the bias- and the variance-term.

ONLINE GRADIENT DESCENT: REGRET

- As a consequence, we can also derive a similar order of the regret for the OGD algorithm on OCO problems as for the FTRL on OLO problems by imposing a slightly different assumption on the (new) “variance” term $\sum_{t=1}^{T} ||\nabla_a(a_t^{\mathtt{OGD}}, z_t)||_2^2$.

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- **Corollary:** Suppose we use the OGD algorithm on an online convex optimization problem with a convex action space $\mathcal{A} \subset \mathbb{R}^d$ such that
  - $\sup_{\tilde{a}\in\mathcal{A}} ||\tilde{a}||_2 \leq B$ for some finite constant $B > 0$
  - $\sup_{a\in\mathcal{A}, z\in\mathcal{Z}} ||\nabla_a(a, z)||_2 \leq V$ for some finite constant $V > 0$.

  Then, by choosing the step size η for OGD as $\eta = \frac{B}{V\sqrt{2T}}$ we get

$$R_T^{\mathtt{OGD}} \leq BV\sqrt{2T}.$$

REGRET LOWER BOUNDS FOR OCO

- **Theorem.** For any online learning algorithm there exists an online convex optimization problem characterized by a convex loss function , a bounded (convex) action space $\mathcal{A} = [-B, B]^d$ and bounded gradients $\sup_{a \in \mathcal{A}, z \in \mathcal{Z}} ||\nabla_a(a, z)||_2 \leq V$ for some finite constants $B, V > 0$, such that the algorithm incurs a regret of $\Omega(\sqrt{T})$ in the worst case.

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- Recall that under (almost) the same assumptions as the theorem above, we have $R_T^{\mathtt{OGD}} \leq BV\sqrt{2\,T}$.

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- Recall that under (almost) the same assumptions as the theorem above, we have $R_T^{\texttt{OGD}} \leq BV\sqrt{2\,T}$.

$\rightsquigarrow$ This result shows that the Online Gradient Descent is *optimal* regarding its order of its regret with respect to the time horizon T.

Outlook

ONLINE MACHINE LEARNING: OUTLOOK

Online machine learning is a very large field of research.

Online Learning			
Statistical Learning Theory	Convex Optimization Theory		Game Theory
Online Learning with Full Feedback		**Online Learning with Partial Feedback (Bandits)**	
Online Supervised Learning		**Stochastic Bandit**	**Adversarial Bandit**
First-order Online Learning	Online Learning with Regularization	Stochastic Multi-armed Bandit	Adversarial Multi-armed Bandit
Second-order Online Learning	Online Learning with Kernels	Bayesian Bandit	Combinatorial Bandit
Prediction with Expert Advice	Online to Batch Conversion	Stochastic Contextual Bandit	Adversarial Contextual Bandit
Applied Online Learning		**Online Active Learning**	**Online Semi-supervised Learning**
Cost-Sensitive Online Learning	Online Collaborative Filtering	Selective Sampling	Online Manifold Regularization
Online Multi-task Learning	Online Learning to Rank	Active Learning with Expert Advice	Transductive Online Learning
Online Multi-view Learning	Distributed Online Learning	**Online Unsupervised Learning (no feedback)**	
Online Transfer Learning	Online Learning with Neural Networks	Online Clustering	Online Density Estimation
Online Metric Learning	Online Portfolio Selection	Online Dimension Reduction	Online Anomaly Detection

Figure: Hoi et al. (2018), ”Online Learning: A Comprehensive Survey”.