ONLINE GRADIENT DESCENT

 $\bullet~$ The *Online Gradient Descent* (OGD) algorithm with step size $\eta>0$ chooses its action by

$$a_{t+1}^{\text{DGD}} = a_t^{\text{DGD}} - \eta \nabla_a(a_t^{\text{DGD}}, z_t), \quad t = 1, \dots T.$$
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(Technical side note: For this update formula we assume that $\mathcal{A}=\mathbb{R}^d$. Moreover, the first action $a_1^{\tt OGD}$ is arbitrary.)



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- We have the following connection between FTRL and OGD:
 - Let $\tilde{z}_t^{\text{DGD}} := \nabla_a(a_t^{\text{DGD}}, z_t)$ for any $t = 1, \dots, T$.
 - The update formula for FTRL with $_2$ norm regularization for the linear loss L^{lin} and the environmental data \tilde{Z}_t^{GD} is

$$a_{t+1}^{ t FTRL} = a_t^{ t FTRL} - \eta ilde{z}_t^{ t GGD} = a_t^{ t FTRL} - \eta
abla_a(a_t^{ t GGD}, z_t).$$

• If we have that $a_1^{\text{FTRL}} = a_1^{\text{OGD}}$, then it iteratively follows that $a_{t+1}^{\text{FTRL}} = a_{t+1}^{\text{OGD}}$ for any $t = 1, \dots, T$ in this case.



ONLINE GRADIENT DESCENT: DEFINITION AND PROPERTIES

With the deliberations above we can infer that

$$\begin{split} R_{T,}^{\text{OGD}}(\tilde{\boldsymbol{a}} \mid (\boldsymbol{z}_t)_t) &= \sum\nolimits_{t=1}^{T} (\boldsymbol{a}_t^{\text{OGD}}, \boldsymbol{z}_t) - (\tilde{\boldsymbol{a}}, \boldsymbol{z}_t) \\ &\leq \sum\nolimits_{t=1}^{T} L^{\text{lin}}(\boldsymbol{a}_t^{\text{OGD}}, \tilde{\boldsymbol{z}}_t^{\text{OGD}}) - L^{\text{lin}}(\tilde{\boldsymbol{a}}, \tilde{\boldsymbol{z}}_t^{\text{OGD}}) \\ & \text{(if } \boldsymbol{a}_1^{\text{OGD}} &= \boldsymbol{a}_1^{\text{FTRL}}) \sum\nolimits_{t=1}^{T} L^{\text{lin}}(\boldsymbol{a}_t^{\text{FTRL}}, \tilde{\boldsymbol{z}}_t^{\text{OGD}}) - L^{\text{lin}}(\tilde{\boldsymbol{a}}, \tilde{\boldsymbol{z}}_t^{\text{OGD}}) \\ &= R_{T, L^{\text{lin}}}^{\text{FTRL}}(\tilde{\boldsymbol{a}} \mid (\tilde{\boldsymbol{z}}_t^{\text{OGD}})_t), \end{split}$$

where we write in the subscripts of the regret the corresponding loss function and also include the corresponding environmental data as a second argument in order to emphasize the connections.



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where we write in the subscripts of the regret the corresponding loss function and also include the corresponding environmental data as a second argument in order to emphasize the connections.

• Interpretation: The regret of the FTRL algorithm (with $_2$ norm regularization) for the online linear optimization problem (characterized by the linear loss L^{lin}) with environmental data \tilde{z}_t^{0GD} is an upper bound for the OGD algorithm for the online convex problem (characterized by a differentiable convex loss) with the original environmental data z_t .



- Due to this connection we immediately obtain a similar decomposition of the regret upper bound into a bias term and a variance term as for the FTRL algorithm for OLO problems.
- Corollary. Using the OGD algorithm on any online convex optimization problem (with differentiable loss function) leads to a regret of OGD with respect to any action $\tilde{a} \in \mathcal{A}$ of

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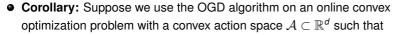
• Note that the step size $\eta > 0$ of OGD has the same role as the regularization magnitude of FTRL: It should balance the trade-off between the bias- and the variance-term.



• As a consequence, we can also derive a similar order of the regret for the OGD algorithm on OCO problems as for the FTRL on OLO problems by imposing a slightly different assumption on the (new) "variance" term $\sum_{t=1}^{T} ||\nabla_a(a_t^{\text{OGD}}, z_t)||_2^2.$



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- $\sup_{\tilde{a} \in A} ||\tilde{a}||_2 \le B$ for some finite constant B > 0
- $\sup_{a \in \mathcal{A}, z \in \mathcal{Z}} ||\nabla_a(a, z)||_2 \le V$ for some finite constant V > 0.

Then, by choosing the step size η for OGD as $\eta = \frac{B}{V\sqrt{2T}}$ we get

$$R_T^{\text{OGD}} \leq BV\sqrt{2T}$$
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- Recall that under (almost) the same assumptions as the theorem above, we have $R_T^{0GD} \leq BV\sqrt{2T}$.
- \rightarrow This result shows that the Online Gradient Descent is *optimal* regarding its order of its regret with respect to the time horizon T.

