Interpretable Machine Learning

Shapley Values

Learning goals

- Learn what game theory is
- Understand the concept behind cooperative games
- Understand the Shapley value in game theory

COOPERATIVE GAMES IN GAME THEORY

Shapley (1951)

- Game theory is the study of strategic games between players, "game" refers to any series of interactions between actors/agents with gains and losses of quantifiable utility value

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COOPERATIVE GAMES IN GAME THEORY

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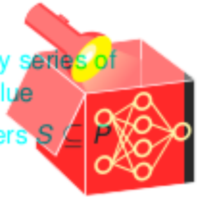

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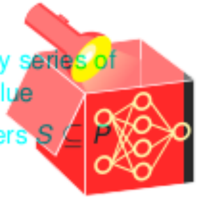

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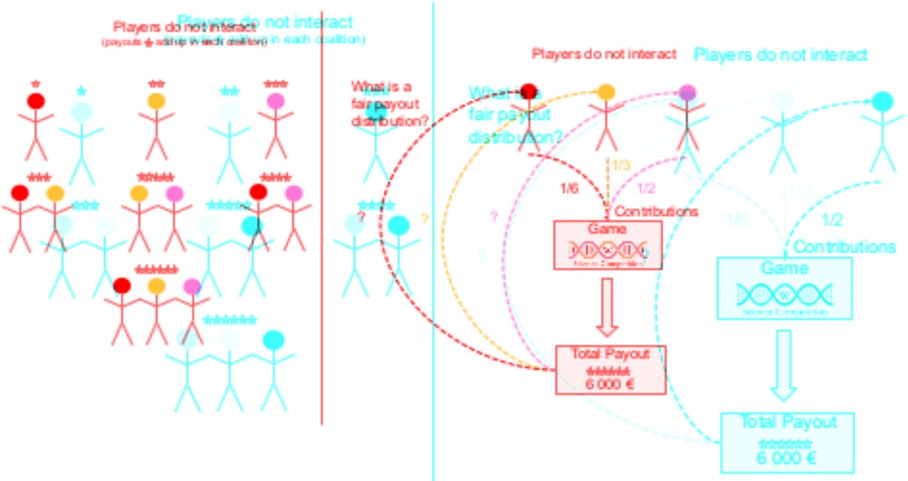
- As some players contribute more than others, we want to fairly divide the total achievable payout $v(P)$ among the players according to a player's individual contribution
- We call the individual payout per player ϕ_j, $j \in P$ (later: Shapley value)

COOPERATIVE GAMES WITHOUT INTERACTIONS



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COOPERATIVE GAMES WITHOUT INTERACTIONS

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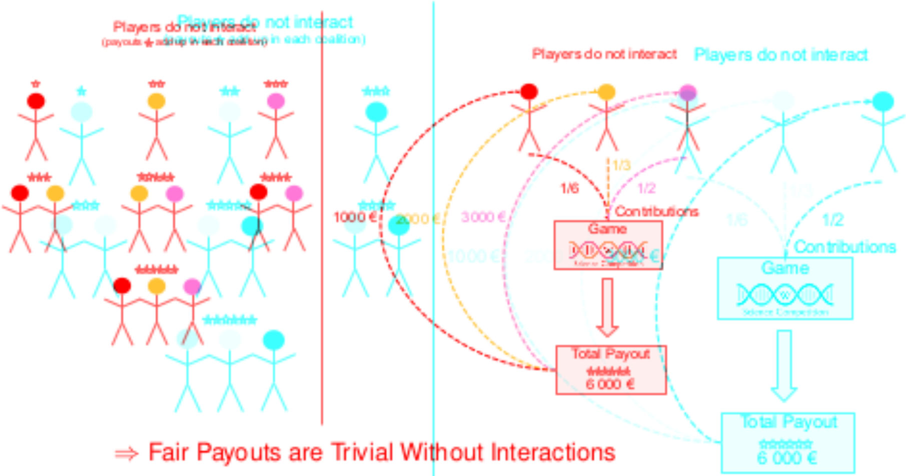

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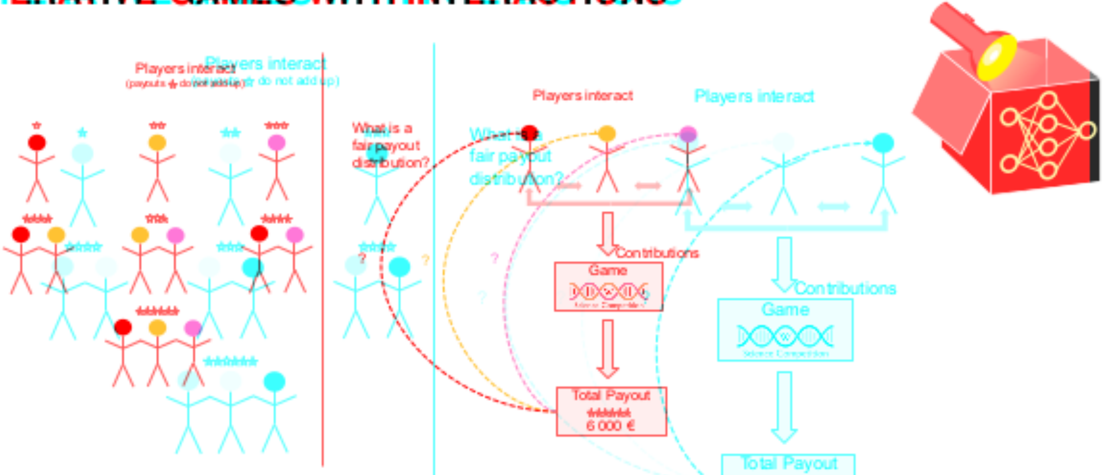
COOPERATIVE GAMES WITH INTERACTIONS



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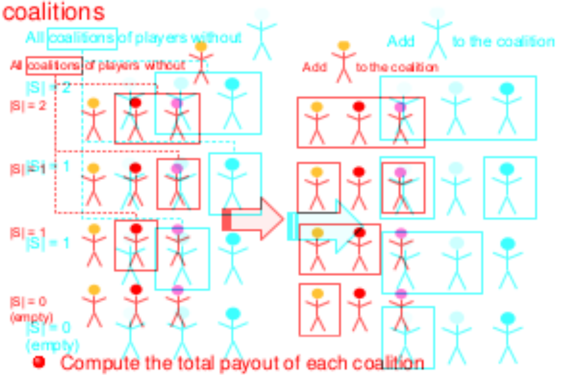
COOPERATIVE GAMES WITH INTERACTIONS

Question: What is a fair payout for player "yellow"?

Idea: Compute marginal contribution of the player of interest across different coalitions



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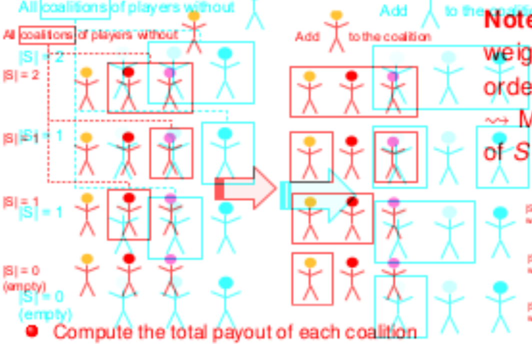
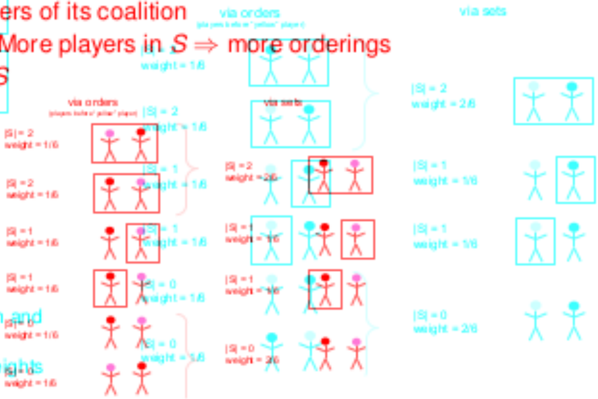
Note: Each marginal contribution is weighted w.r.t. number of possible orders of its coalition

⇝ More players in $S \Rightarrow$ more orderings of S





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SHAPLEY VALUE - SET DEFINITION

This idea refers to the **Shapley value** which assigns a payout value to each player according to its marginal contribution in all possible coalitions.

- Let $v(S \cup \{j\}) - v(S)$ be the marginal contribution of player j to coalition S
  $\leadsto$ measures how much a player j increases the value of a coalition S

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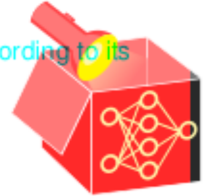

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- Shapley value via **set definition** (weighting via multinomial coefficient):

$$\phi_j = \sum_{S \subseteq P \setminus \{j\}} \frac{|S|!(|P| - |S| - 1)!}{|P|!} (v(S \cup \{j\}) - v(S))$$

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SHAPLEY VALUE - ORDER DEFINITION

The Shapley value was introduced as summation over sets $S \subseteq P \setminus \{j\}$, but it can be equivalently defined as a summation of all orders of players:

$$\phi_j = \frac{1}{|P|!} \sum_{\tau \in \Pi} \left(v(S_j^\tau \cup \{j\}) - v(S_j^\tau) \right)$$

- Π: All possible orders of players (we have $|P|!$ in total)

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- S_j^τ: Set of players before player j in order $\tau = (\tau^{(1)}, \dots, \tau^{(p)})$ where $\tau^{(i)}$ is i-th element
  $\Rightarrow$ Example: Players $1, 2, 3 \Rightarrow \Pi = \{(1, 2, 3), (1, 3, 2), (2, 1, 3), (2, 3, 1), (3, 1, 2), (3, 2, 1)\}$
  - $\rightsquigarrow$ For order $\tau = (2, 1, 3)$ and player of interest $j = 3 \Rightarrow S_j^\tau = \{2, 1\}$
  - $\rightsquigarrow$ For order $\tau = (3, 1, 2)$ and player of interest $j = 1 \Rightarrow S_j^\tau = \{3\}$
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- Order definition: Marginal contribution of orders that yield set $S = \{1, 2\}$ is summed twice
  $\rightsquigarrow$ In set definition, it has the weight $\frac{2!(3-2-1)!}{3!} = \frac{2 \cdot 0!}{6} = \frac{2}{6}$

SHAPLEY VALUE - COMMENTS ON ORDER DEFINITION

- Order and set definition are equivalent
- Reason: The number of orders which yield the same coalition S is $|S|!(|P| - |S| - 1)!$
  - $\Rightarrow$ There are $|S|!$ possible orders of players within coalition S
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$|S|!$ permutations — $(|P| - |S| - 1)!$ permutations

$\tau^{(1)}$	$\dots$	$\tau^{(\|S\|)}$	$\tau^{(\|S\|+1)}$	$\tau^{(\|S\|+2)}$	$\dots$	$\tau^{(p)}$

Players before player j — player j — Players after player j

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$\lvert S \rvert!$ permutations				$(\lvert P \rvert - \lvert S \rvert - 1)!$ permutations		
Players before player j			player j	Players after player j		

- Relevance of the order definition: Approximate Shapley values by sampling permutations
  $\rightsquigarrow$ randomly sample a fixed number of M permutations and average them:

$$\phi_j = \frac{1}{M} \sum_{\tau \in \Pi_M} \left(v(S_j^\tau \cup \{j\}) - v(S_j^\tau) \right)$$

where $\Pi_M \subset \Pi$ is a random subset of Π containing only M orders of players

WEIGHTS FOR MARGINAL CONTRIBUTION - ILLUSTRATION

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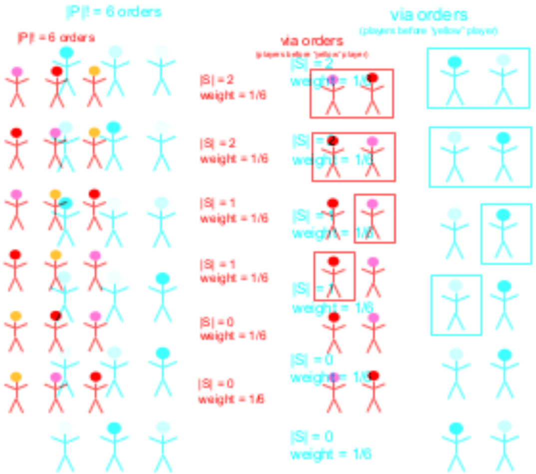

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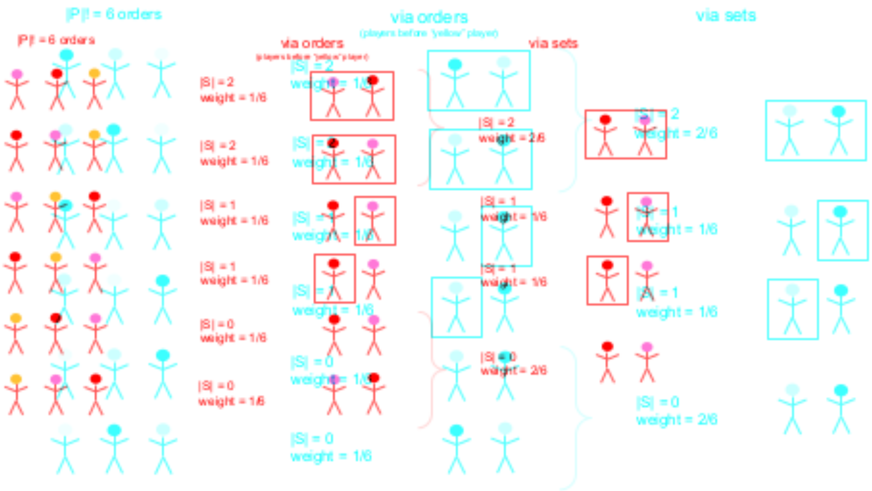
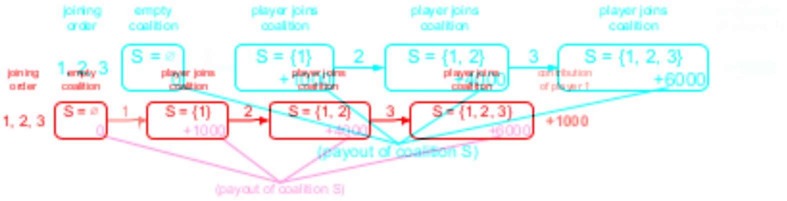


SHAPLEY VALUES - ILLUSTRATION

- Shapley value of player j is the marginal contribution to the value when it enters any coalition
- Produce all possible joining orders of player coalitions



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SHAPLEY VALUES - ILLUSTRATION

- Shapley value of player j is the marginal contribution to the value when it enters any coalition
- Produce all possible joining orders of player coalitions
- Measure and average the difference in payout after player 1 enters the coalition

joining order	empty coalition	player joins coalition	player joins coalition	player joins coalition	contribution of player 1
1, 2, 3	S = ∅ 0	1 → S = {1} +1000	2 → S = {1, 2} +4000	3 → S = {1, 2, 3} +6000	+1000
1, 3, 2	S = ∅ 0	1 → S = {1} +1000	3 → S = {1, 3} +4000	2 → S = {1, 2, 3} +6000	+1000
2, 1, 3	S = ∅ 0	2 → S = {2} +2000	1 → S = {1, 2} +4000	3 → S = {1, 2, 3} +6000	+2000
2, 3, 1	S = ∅ 0	2 → S = {2} +2000	3 → S = {2, 3} +3000	1 → S = {1, 2, 3} +6000	+3000
3, 1, 2	S = ∅ 0	3 → S = {3} +3000	1 → S = {1, 3} +4000	2 → S = {1, 2, 3} +6000	+1000
3, 2, 1	S = ∅ 0	3 → S = {3} +3000	2 → S = {2, 3} +3000	1 → S = {1, 2, 3} +6000	+3000

Shapley value of player 1: +1833.33

SHAPLEY VALUES - ILLUSTRATION

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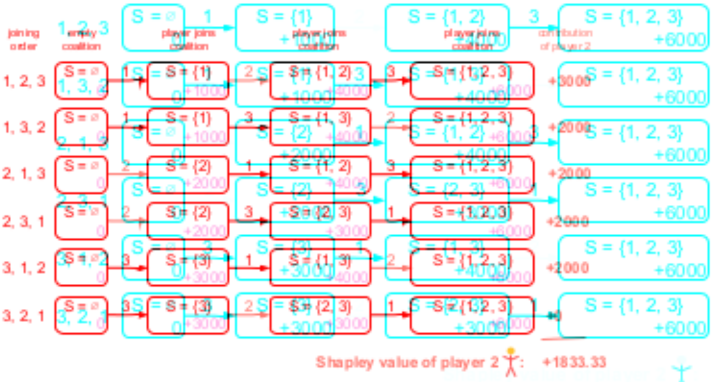
- Produce all possible joining orders of player coalitions
- Measure and average the difference in payout after player 2 enters the coalition



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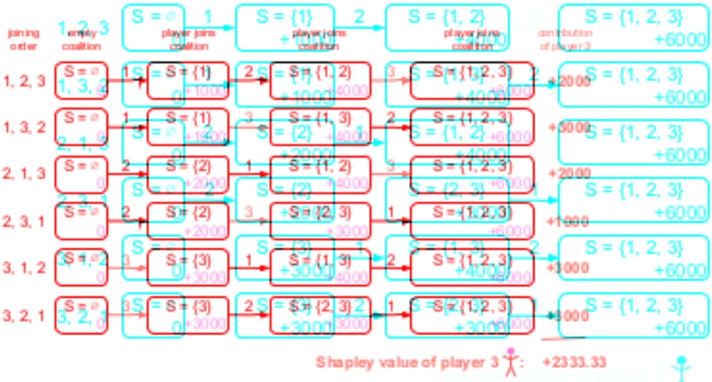
- Measure and average the difference in payout after player 3 enters the coalition



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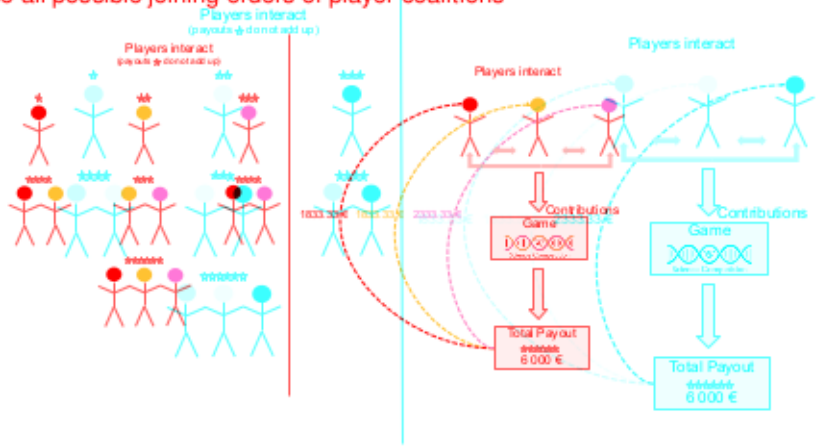

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AXIOMS OF FAIR PAYOUTS

Why is this a fair payout solution?

One possibility to define fair payouts are the following axioms for a given value function v:

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  If $v(S \cup \{j\}) = v(S \cup \{k\})$ for all $S \subseteq P \setminus \{j, k\}$, then $\phi_j = \phi_k$

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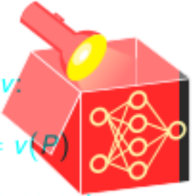

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- **Additivity**: For a game v with combined payouts $v(S) = v_1(S) + v_2(S)$, the payout is the sum of payouts: $\phi_{j,v} = \phi_{j,v_1} + \phi_{j,v_2}$

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