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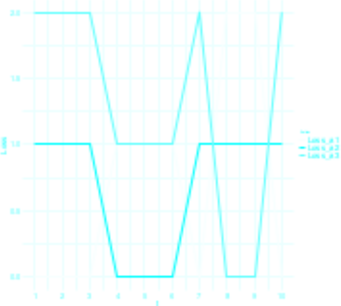
Advanced Machine Learning

Simple Online Learning Algorithms



Learning goals

- Formalization of online learning algorithms
- Getting to know the FTL algorithm
- See that it works for online quadratic optimzation (OQO) problems

THE ONLINE LEARNER

- In the following, we will consider a first (online) learner for online learning problems. Note that a learner can be defined in a formal way.
- Indeed, a learner within the basic online learning protocol, say `Algo`, is a function

$$A : \bigcup_{t=1}^{T} (\mathcal{Z} \times \mathcal{A})^t \to \mathcal{A}$$

that returns the current action based on (the loss and) the full history of information so far:

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- In the extended online learning scenario, where the environmental data consists of two parts, $z_t = (z_t^{(1)}, z_t^{(2)})$, and the first part is revealed before the action in t is performed, we have that

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- It will be desired that the online learner admits a *cheap update formula*, which is incremental, i.e., only a portion of the previous data is necessary to determine the next action.
- For instance, there exists a function $u : \mathcal{Z} \times \mathcal{A} \to \mathcal{A}$ such that

$$A(z_1, a_1^{\texttt{Algo}}, z_2, a_2^{\texttt{Algo}}, \ldots, z_t, a_t^{\texttt{Algo}};) = u(z_t, a_t^{\texttt{Algo}}).$$

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FOLLOW THE LEADER ALGORITHM

- A simple algorithm to tackle online learning problems is the **Follow the leader** (FTL) algorithm.
- The algorithm takes as its action $a_t^{\text{FTL}} \in \mathcal{A}$ in time step $t \geq 2$, the element which has the minimal cumulative loss so far over the previous $t-1$ time periods:

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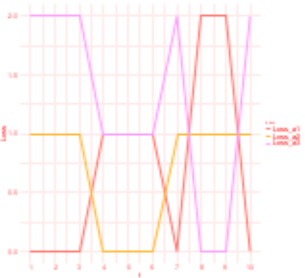
- *Interpretation:* The action a_t^{FTL} is the current "leader" of the actions in $\mathcal{A}$ in time step t, as it has the smallest cumulative loss (error) so far.



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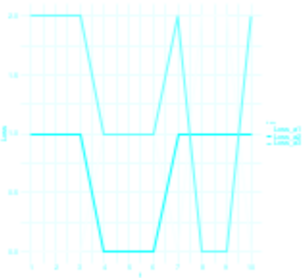

FTL: A HELPFUL LEMMA

Lemma: Let $a_1^{\mathrm{FTL}}, a_2^{\mathrm{FTL}}, \ldots$ be the sequence of actions used by the FTL algorithm for the environmental data sequence $z_1, z_2, \ldots$.

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R_T^{\mathrm{FTL}}(\tilde{a}) &= \sum_{t=1}^{T} \left((a_t^{\mathrm{FTL}}, z_t) - (\tilde{a}, z_t) \right) \\
&\leq \sum_{t=1}^{T} \left((a_t^{\mathrm{FTL}}, z_t) - (a_{t+1}^{\mathrm{FTL}}, z_t) \right) \\
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Interpretation: the regret of the FTL algorithm is bounded by the difference of cumulated losses of itself compared to its one-step lookahead cheater version.

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Reminder: $a_t^{\mathrm{FTL}} \in \arg\min_{a \in \mathcal{A}} \sum_{s=1}^{t-1} (a, z_s).$

Initial step: $T = 1$. It holds that

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$$\sum_{t=1}^{T} (a_{t+1}, z_t) \leq (a_{T+1}, z_T) + \sum_{t=1}^{T-1} (\tilde{a}, z_t), \quad \forall \tilde{a} \in \mathcal{A}.$$

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FTL FOR OQO PROBLEMS

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- In its most general form, the loss function is thereby defined as

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- **Proposition:** Using FTL on any online quadratic optimization problem with $\mathcal{A} = \mathbb{R}^d$ and $V = \sup_{z \in \mathcal{Z}} ||z||_2$, leads to a regret of

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