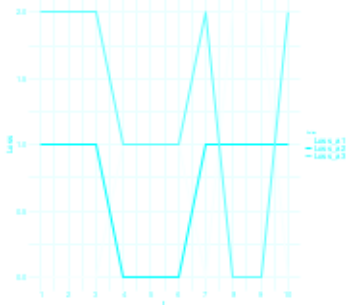


FTL FOR ONLINE LINEAR OPTIMIZATION

Advanced Machine Learning

- Another popular instantiation of the online learning problem is the online linear optimization problem, which is characterized by a linear loss function $(a, z) = a^T z$.

Follow the leader on OLO problems



Learning goals

- Getting to know online linear optimization (OLO) problems
- See that FTL might fail for these problems
- Understanding the root cause for FTL's flaw

FTL FOR ONLINE LINEAR OPTIMIZATION

- Another popular instantiation of the online learning problem is the online linear optimization problem, which is characterized by a linear loss function $\ell(a, z) = a^T z$.

- Let $\mathcal{A} = [-1, 1]$ and suppose that $z_t = \begin{cases} -\frac{1}{2}, & t = 1, \\ 1, & t \text{ is even,} \\ -1, & t \text{ is odd.} \end{cases}$



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- Another popular instantiation of the online learning problem is the online linear optimization problem, which is characterized by a linear loss function $\ell(a, z) = -a^T z$.

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- No matter how we choose the first action a_1^{FTL} , it will hold that FTL has a cumulative loss greater than (or equal) $T - 3/2$, while the best action in hindsight has a cumulative loss of $-1/2$.
- Thus, FTL's cumulative regret is at least $T - 1$, which is linearly growing in T .



FTL FOR ONLINE LINEAR OPTIMIZATION

• Indeed, note that

• Another popular instantiation of the online learning problem is the online linear optimization problem, which is characterized by a linear loss function $L(a, z) \equiv a \cdot z$

$$a_1^{\text{FTL}} \equiv \arg \min_{a \in \mathcal{A}} \sum_{s=1}^T (a, z_s) = \arg \min_{a \in [-1, 1]} a \sum_{s=1}^T z_s$$

• Let $\mathcal{A} = [-1, 1]$ and suppose that $\bar{z}_t = \begin{cases} -1, & \text{if } \sum_{s=1}^t z_s > 0, \\ 1, & \text{if } \sum_{s=1}^t z_s < 0, \\ \text{arbitrary}, & \text{if } \sum_{s=1}^t z_s = 0. \end{cases}$

- No matter how we choose the first action a_1^{FTL} , it will hold that FTL has a cumulative loss greater than (or equal) $T - 3/2$, while the best action in hindsight has a cumulative loss of $-1/2$.
- Thus, FTL's cumulative regret is at least $T - 1$, which is linearly growing in T .



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- Indeed, note that

$$\begin{aligned}
 a_t^{\text{FTL}} a_{t+1}^{\text{FTL}} &= \arg \min_{a \in \mathcal{A}} \sum_{s=1}^t (a, z_s) z_s = \arg \min_{a \in [-1, 1]} \sum_{s=1}^t z_s \cdot a = \arg \min_{a \in [-1, 1]} a \sum_{s=1}^t z_s \\
 &= \begin{cases} -1, & \text{if } \sum_{s=1}^t z_s \geq 0, \\ 1, & \text{if } \sum_{s=1}^t z_s \leq 0, \\ \text{arbitrary}, & \text{if } \sum_{s=1}^t z_s \approx 0. \end{cases}
 \end{aligned}$$



t	a_t^{FTL}	z_t	(a_t^{FTL}, z_t)	$\sum_{s=1}^t (a_s^{\text{FTL}}, z_s)$	$\sum_{s=1}^t z_s$
1	1	-1/2	-1/2	-1/2	-1/2

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$$\begin{aligned}
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 &= \begin{cases} -1, -1, & \text{if } \sum_{s=1}^t z_s \geq 0; 0, \\ 1, 1, & \text{if } \sum_{s=1}^t z_s \leq 0; 0, \\ \text{arbitrary,} & \text{if } \sum_{s=1}^t z_s \approx 0; 0. \end{cases}
 \end{aligned}$$



t	a_t^{FTL}	z_t	(a_t^{FTL}, z_t)	$\sum_{s=1}^t (a_s^{\text{FTL}}, z_s)$	$\sum_{s=1}^t z_s$
1	1	-1/2	-1/2	-1/2	-1/2
2	1	1	1	1 - 1/2	1/2

FTL FOR ONLINE LINEAR OPTIMIZATION

- Indeed, note that

$$\begin{aligned}
 a_t^{\text{FTL}} a_{t+1}^{\text{FTL}} &= \arg \min_{a \in \mathcal{A}} \sum_{s=1}^t (a \cdot z_s) z_s = \arg \min_{a \in [-1,1]} \sum_{s=1}^t z_s \cdot [1] a \sum_{s=1}^t z_s \\
 &= \begin{cases} -1, -1, & \text{if } \sum_{s=1}^t z_s \geq 0; 0, \\ 1, 1, & \text{if } \sum_{s=1}^t z_s \leq 0; 0, \\ \text{arbitrary,} & \text{if } \sum_{s=1}^t z_s \approx 0; 0. \end{cases}
 \end{aligned}$$



t	a_t^{FTL}	z_t	(a_t^{FTL}, z_t)	$\sum_{s=1}^t (a_s^{\text{FTL}} z_s)$	$\sum_{s=1}^t z_s z_s$
1	1	-1/2	-1/2	-1/2/2	-1/2/2
2	1	1	1	1 + 1/2/2	1/2/2
3	-1	-1	1	2 - 1/2	-1/2

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- Indeed, note that

$$\begin{aligned}
 a_t^{\text{FTL}} a_{t+1}^{\text{FTL}} &= \arg \min_{a \in \mathcal{A}} \sum_{s=1}^t (a \cdot z_s) z_s = \arg \min_{a \in [-1,1]} \sum_{s=1}^t z_s \cdot [a] a \sum_{s=1}^t z_s \\
 &= \begin{cases} -1, -1, & \text{if } \sum_{s=1}^t z_s \geq 0; 0, \\ 1, 1, & \text{if } \sum_{s=1}^t z_s \leq 0; 0, \\ \text{arbitrary,} & \text{if } \sum_{s=1}^t z_s \approx 0; 0. \end{cases}
 \end{aligned}$$



t	a_t^{FTL}	z_t	(a_t^{FTL}, z_t)	$\sum_{s=1}^t (a_s^{\text{FTL}} z_s)$	$\sum_{s=1}^t z_s z_s$
1	1	-1/2	-1/2	-1/2/2	-1/2/2
2	1	1	1	1 + 1/2/2	1/2/2
3	-1	-1	1	2 - 1/2/2	-1/2/2
\vdots	\vdots	\vdots	\vdots	\vdots	\vdots
T	$(-1)^T$	$(-1)^T$	1	$T - 1 - 1/2$	$(-1/2)^T$

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- Indeed, note that

$$\begin{aligned}
 a_t^{\text{FTL}} &= \arg \min_{a \in \mathcal{A}} \sum_{s=1}^t (a, z_s) z_s = \arg \min_{a \in [-1,1]} \sum_{s=1}^t z_{s,1} a \sum_{s=1}^t z_s \\
 &= \begin{cases} -1, -1, & \text{if } \sum_{s=1}^t z_s \geq 0, \\ 1, 1, & \text{if } \sum_{s=1}^t z_s \leq 0, \\ \text{arbitrary}, & \text{if } \sum_{s=1}^t z_s \approx 0. \end{cases}
 \end{aligned}$$



t	a_t^{FTL}	z_t	(a_t^{FTL}, z_t)	$\sum_{s=1}^t (a_s^{\text{FTL}}, z_s)$	$\sum_{s=1}^t z_s$
1	1	-1/2	-1/2	-1/2	-1/2
2	1	1	1	1 + 1/2	1/2
3	-1	-1	-1	2 - 1/2	-1/2
\vdots	\vdots	\vdots	\vdots	\vdots	\vdots
T	$(-1)^T$	$(-1)^T$	T	$T - 1 + 1/2$	$(-1/2)^T$

- The best action has cumulative loss

$$\inf_{a \in \mathcal{A}} \sum_{s=1}^T (a, z_s) = \inf_{a \in [-1,1]} a \underbrace{\sum_{s=1}^T z_s}_{= (-1/2)^T} = -1/2.$$

FTL FOR ONLINE LINEAR OPTIMIZATION

- Thus, we see: FTL can fail for online linear optimization problems, although it is well suited for online quadratic optimization problems!
- The reason is that the action selection of FTL is not stable enough (caused by the loss function), which is fine for the latter problem, but problematic for the former.
- One has to note that the online linear optimization problem example above, where FTL fails, is in fact an adversarial learning setting: The environmental data is generated in such a way that the FTL learner is fooled in each time step.



t	a_t	z_t	$\sum_{s=1}^t z_s$	a_t^{FTL}	$\sum_{s=1}^t z_s^{\text{FTL}}$
1	1	-1/2	-1/2	-1/2	-1/2
2	1	1	1	1	1/2
3	-1	-1	0	1	-1/2
\vdots	\vdots	\vdots	\vdots	\vdots	\vdots
T	$(-1)^T$	$(-1)^T$	1	$T-1$	$(-1/2)^T$

- The best action has cumulative loss

$$\inf_{a \in \mathcal{A}} \sum_{s=1}^T L(a, z_s) = \inf_{a \in [-1, 1]} a \underbrace{\sum_{s=1}^T z_s}_{=(-1/2)^T} = -1/2.$$

FTL FOR ONLINE LINEAR OPTIMIZATION

- Thus, we see: FTL can fail for $\text{minimize}_{\mathbf{w}} \sum_{t=1}^T \ell_t(\mathbf{w})$ although it is well suited for online quadratic optimization problems!
- The reason is that the action selection of FTL is not stable enough (caused by the loss function), which is fine for the latter problem, but problematic for $\text{minimize}_{\mathbf{w}} \sum_{t=1}^T \ell_t(\mathbf{w})$.
- One has to note that the online linear optimization problem example above, where FTL fails, is in fact an adversarial learning setting: The environmental data is generated in such a way that the FTL learner is fooled in each time step.

