BATCH LEARNING

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  - A data set $\mathcal{D} = \{(\mathbf{x}^{(i)}, y^{(i)})\}_{i=1}^{n}$ is given beforehand in form of a random sample (iid observations).

Advanced Machine Learning

Introduction to Online Learning

Learning goals

- Understand the difference between batch and online learning
- Know the basic and the extended learning protocol in online learning
- Know how performance is measured in online learning

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- The learning task on the available data beforehand is called the *training phase* and the prediction on the unseen data is called the *testing phase*. Both phases are **separated**.

ONLINE LEARNING

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  - *Sequential order* — data is generated only bit by bit;
  - *On-the-fly decisions* — decisions or predictions have to be made during the data generating process;
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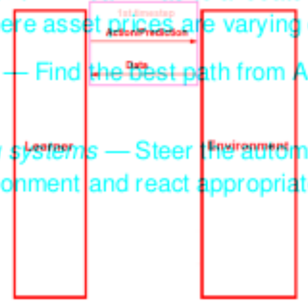

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THE BASIC ONLINE LEARNING PROTOCOL

Formally, an online learning problem consists of:

- a learner (forecaster, agent resp. decision maker), an environment (user resp. adversary, system resp. nature),
- time steps $1, 2, \dots, T$ (may be infinite),
- available actions $\mathcal{A}$ for the learner (may be infinite),
- environmental data space $\mathcal{Z}$,
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Typically $\mathcal{A} = \mathcal{Z} = \mathcal{Y}$, so that

- the learner's chosen action $a_t = \hat{y}_t$ corresponds to a prediction,
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- This also covers the area of *adversarial learning*: the data can even be generated by an adversary trying to fool the learner.
- However, the online learning setting can of course also be considered in a statistical setting.

ONLINE LEARNING: REQUIREMENTS

- The dynamical aspects have to be incorporated for the design of efficient learning algorithms.
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- We will mainly focus our theoretical analysis on the performance of the learner in terms of its (cumulative) loss, which, however, will usually ignore computational aspects of the learner.

MEASURE OF QUALITY IN ONLINE LEARNING

- In order to measure the quality of an online learner one can compute the difference between the cumulative loss of the learner and the cumulative loss by taking some competing action $a \in \mathcal{A}$:

$$R_T(a) = \sum_{t=1}^{T} L(a_t, z_t) - \sum_{t=1}^{T} L(a, z_t).$$

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- Formally, the following should hold

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Interpretation: The average regret per time step (or per example) goes to zero:

$$\frac{1}{T}\left(\sum_{t=1}^{T} L(a_t, z_t) - \inf_{a \in \mathcal{A}} \sum_{t=1}^{T} L(a, z_t)\right) = \frac{R_T}{T} = o(1).$$

DYNAMIC REGRET

- One might ask why one compares only with a fixed best action in hindsight, say a^*, instead of a sequence of actions $a_1^*, a_2^*, \ldots, a_T^*$?
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- However, this is too optimistic and may not hold in changing environments, where data are evolving and the optimal action is drifting over the time.
- To address this limitation, recent works have also considered the *dynamic regret*:

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DYNAMIC REGRET

- One might ask why one compares only with a fixed best action in hindsight, say a^*, instead of a sequence of actions $a_1^*, a_2^*, \ldots, a_T^*$?
- The rationale behind this measure of quality is that the best fixed action in hindsight is already reasonably good over all the time steps: it performs almost as well as a batch learner that observes the entire sequence and picks the best action in hindsight.
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