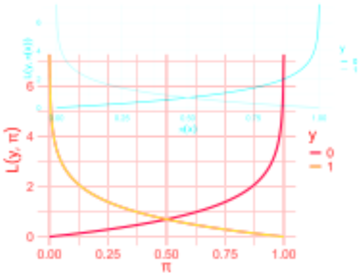
Introduction to Machine Learning

Advanced Risk Minimization

Optimal constant model for the empirical log loss risk (Deep-Dive)



Learning goals

- Derive the optimal constant model for the binary empirical log loss risk
- Derive the optimal constant model for the empirical multiclass log loss risk

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BINARY LOG LOSS: EMP. RISK MINIMIZER / 2

The minimizer can be found by setting the derivative to zero, i.e.,

$$
\begin{aligned}
\frac{d}{d\theta}\mathcal{R}_{\text{emp}} &= -\sum_{i=1}^{n} \frac{y^{(i)}}{\theta} - \frac{1-y^{(i)}}{1-\theta} \overset{!}{=} 0 \\
\iff & -\sum_{i=1}^{n} y^{(i)}(1-\theta) - \theta(1-y^{(i)}) \overset{!}{=} 0 \\
\iff & -\sum_{i=1}^{n} \left(y^{(i)} - \theta\right) \overset{!}{=} 0
\end{aligned}
$$

$\Rightarrow \hat{\theta} = \frac{1}{n}\sum_{i=1}^{n} y^{(i)} \in (0, 1)$ ✓ (assuming both labels occur).

MULTICLASS LOG LOSS: EMP. RISK MINIMIZER / 2

With this, we find the equivalent optimization problem

$$
\begin{aligned}
\arg\min_{\theta \in (0,1)^{g-1}} \mathcal{R}_{\text{emp}} \quad = \quad & \arg\min_{\theta \in (0,1)^{g-1}} - \sum_{i=1}^{n} \sum_{j=1}^{g-1} \mathbb{1}_{\{y^{(i)}=j\}} \log(\theta_j) \\
& + \mathbb{1}_{\{y^{(i)}=g\}} \log\left(1 - \sum\nolimits_{j=1}^{g-1} \theta_j\right) \\
\text{s.t.} \quad & \sum\nolimits_{j=1}^{g-1} \theta_j < 1.
\end{aligned}
$$

For $j \in \{1, \ldots, g-1\}$, the j-th partial derivative of our objective

$$
\begin{aligned}
\frac{\partial}{\partial \theta_j} \mathcal{R}_{\text{emp}} \quad &= \quad -\sum_{i=1}^{n} \mathbb{1}_{\{y^{(i)}=j\}} \frac{1}{\theta_j} - \mathbb{1}_{\{y^{(i)}=g\}} \frac{1}{1 - \sum_{j=1}^{g-1} \theta_j} \\
&= \quad -\frac{n_j}{\theta_j} + \frac{n_g}{\theta_g}
\end{aligned}
$$

where n_k with $k \in \{1, \ldots, g\}$ is the number of label k in y and we assume that $n_k > 0$.

MULTICLASS LOG LOSS: EMP. RISK MINIMIZER / 3

For the minimizer, it must hold for $j \in \{1, \dots, g-1\}$ that

$$
\begin{aligned}
\frac{\partial}{\partial \theta_j} \mathcal{R}_{\text{emp}} &\overset{!}{=} 0 \\
\iff -n_j \theta_g + n_g \theta_j &\overset{!}{=} 0 \\
\Rightarrow \sum_{j=1}^{g-1} (-n_j \theta_g + n_g \theta_j) &\overset{!}{=} 0 \\
\iff -(n - n_g)\theta_g + n_g(1 - \theta_g) &\overset{!}{=} 0 \\
\iff -n\theta_g + n_g &\overset{!}{=} 0 \\
\Rightarrow \hat{\theta}_g = \frac{n_g}{n} & \quad \in (0,1) \checkmark \\
\Rightarrow \forall j \in \{1, \dots, g-1\}: \quad \hat{\theta}_j = \frac{\hat{\theta}_g n_j}{n_g} = \frac{n_j}{n} & \quad \in (0,1) \checkmark .
\end{aligned}
$$

$$
\left(\Rightarrow \sum\nolimits_{j=1}^{g-1} \hat{\theta}_j = 1 - \hat{\theta}_g = 1 - \frac{n_g}{n} < 1 \checkmark \right)
$$