Introduction to Machine Learning

Advanced Risk Minimization

0-1-Loss

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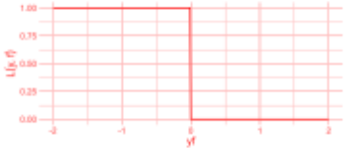

Learning goals

- Derive the risk minimizer of the 0-1-loss
- Derive the optimal constant model for the 0-1-loss

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0-1-LOSS

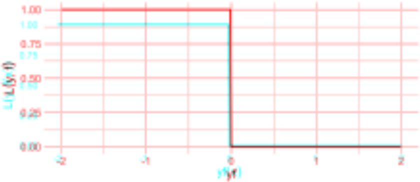
- Let us first consider a discrete classifier $h(\mathbf{x}) : \mathcal{X} \to \mathcal{Y}$.
- The most natural choice for $L(y, h(\mathbf{x}))$ is the 0-1-loss

$$L(y, h(\mathbf{x})) = \mathbb{1}_{\{y \neq h(\mathbf{x})\}} = \begin{cases} 1 & \text{if } y \neq h(\mathbf{x}) \\ 0 & \text{if } y = h(\mathbf{x}) \end{cases}$$

- For the binary case ($g = 2$) we can express the 0-1-loss for a scoring classifier $f(\mathbf{x})$ based on the margin $\nu := yf(\mathbf{x})$

$$L(y, f(\mathbf{x})) = \mathbb{1}_{\{\nu < 0\}} = \mathbb{1}_{\{yf(\mathbf{x}) < 0\}}.$$

- Analytic properties: Not continuous, even for linear f the optimization problem is NP-hard and close to intractable.



0-1-LOSS: RISK MINIMIZER

By the law of total expection we can in general rewrite the risk as (this all works for the multiclass case with 0-1)

$$\begin{aligned}\mathcal{R}(f) &= \mathbb{E}_{xy}\left[L\left(y, f(\mathbf{x})\right)\right] = \mathbb{E}_x\left[\mathbb{E}_{y|x}\left[L\left(y, f(\mathbf{x})\right)\right]\right] \\ &= \mathbb{E}_x\left[\sum_{k \in \mathcal{Y}} L(k, f(\mathbf{x}))\mathbb{P}(y = k \mid \mathbf{x})\right],\end{aligned}$$

with $\mathbb{P}(y = k|\mathbf{x})$ the posterior probability for class k. For the binary case we denote $\eta(\mathbf{x}) := \mathbb{P}(y = 1 \mid \mathbf{x})$ and the expression becomes

$$\mathcal{R}(f) = \mathbb{E}_x\left[L(1, \pi(\mathbf{x})) \cdot \eta(\mathbf{x}) + L(0, \pi(\mathbf{x})) \cdot (1 - \eta(\mathbf{x}))\right].$$