Interpretable Machine Learning

Shapley Values

Learning goals

- Learn what game theory is
- Understand the concept behind cooperative games
- Understand the Shapley value in game theory

COOPERATIVE GAMES IN GAME THEORY

Shapley (1951)

- Game theory is the study of strategic games between players, “game” refers to any series of interactions between actors/agents with gains and losses of quantifiable utility value

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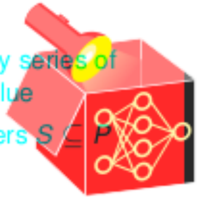
COOPERATIVE GAMES IN GAME THEORY

▸ Shapley (1951)



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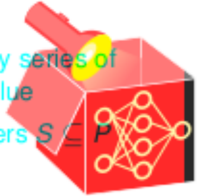

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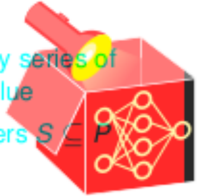

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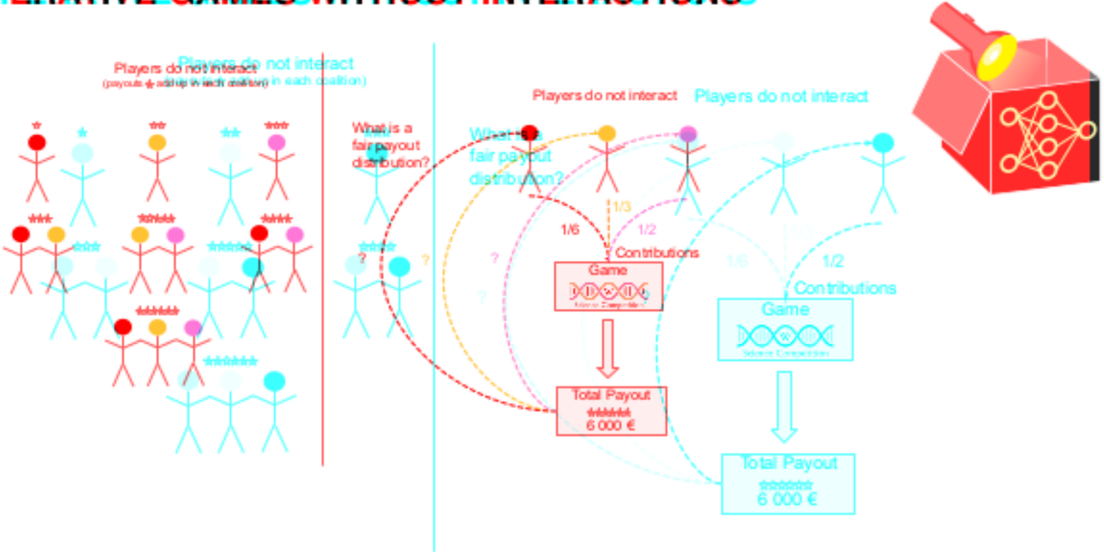
- As some players contribute more than others, we want to fairly divide the total achievable payout $v(P)$ among the players according to a player's individual contribution
- We call the individual payout per player ϕ_j, $j \in P$ (later: Shapley value)

COOPERATIVE GAMES WITHOUT INTERACTIONS



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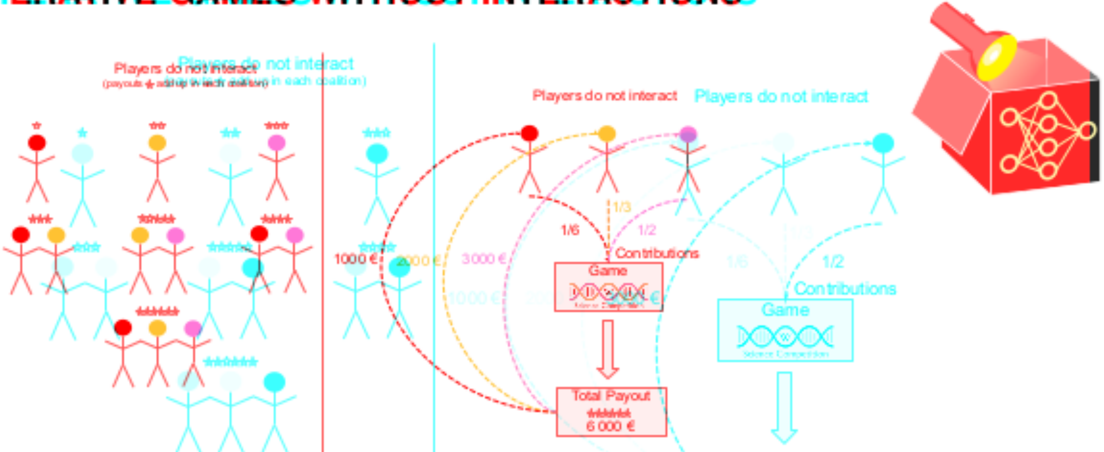
COOPERATIVE GAMES WITHOUT INTERACTIONS



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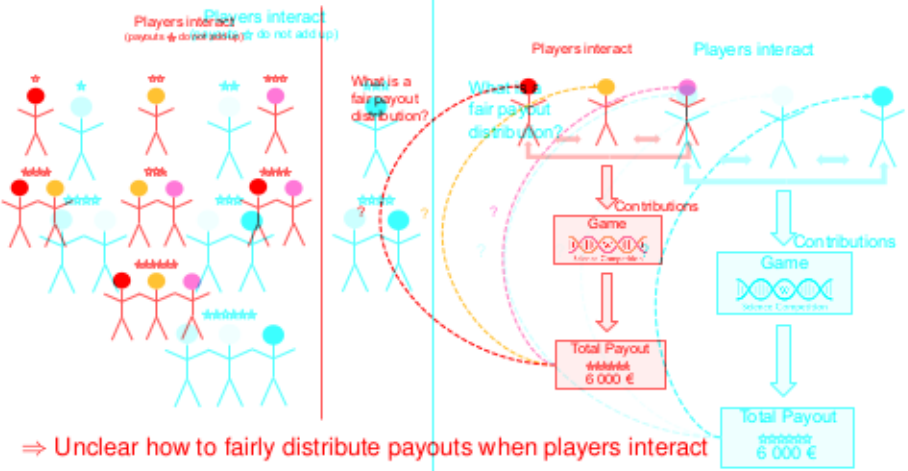
COOPERATIVE GAMES WITH INTERACTIONS



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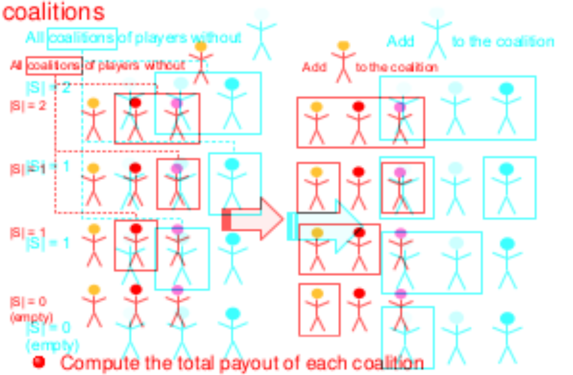
COOPERATIVE GAMES WITH INTERACTIONS

Question: What is a fair payout for player "yellow"?

Idea: Compute marginal contribution of the player of interest across different coalitions



- Compute the total payout of each coalition
- Compute difference in payouts for each coalition with and without player "yellow" (= marginal contribution)
- Average marginal contributions using appropriate weights

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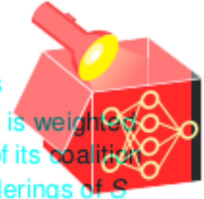
COOPERATIVE GAMES WITH INTERACTIONS



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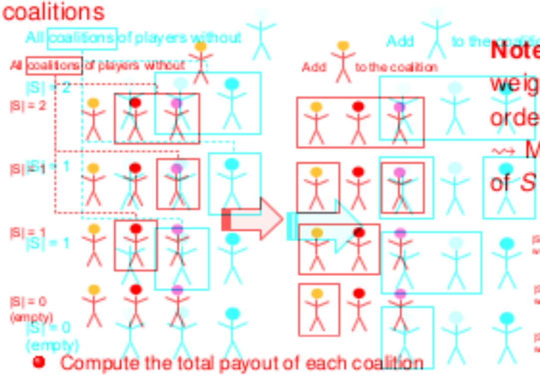
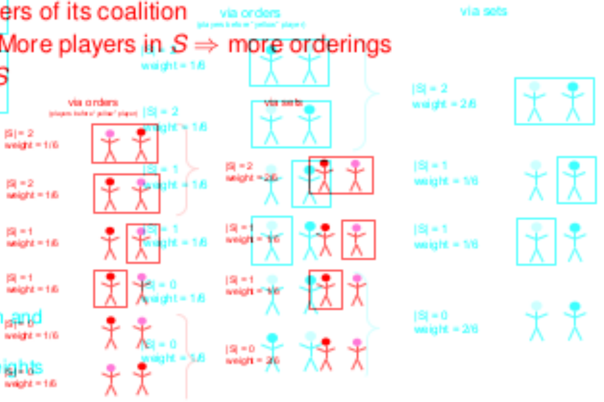


Note: Each marginal contribution is weighted w.r.t. number of possible orders of its coalition

⇝ More players in S ⇒ more orderings of S



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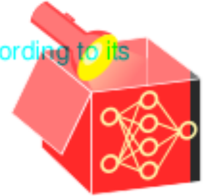
SHAPLEY VALUE - SET DEFINITION



This idea refers to the **Shapley value** which assigns a payout value to each player according to its marginal contribution in all possible coalitions.

- Let $v(S \cup \{j\}) - v(S)$ be the marginal contribution of player j to coalition S
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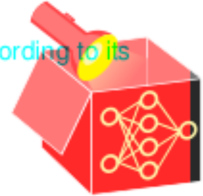

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- Shapley value via **set definition** (weighting via multinomial coefficient):

$$\phi_j = \sum_{S \subseteq P \setminus \{j\}} \frac{|S|!(|P| - |S| - 1)!}{|P|!} (v(S \cup \{j\}) - v(S))$$

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SHAPLEY VALUE - ORDER DEFINITION

The Shapley value was introduced as summation over sets $S \subseteq P \setminus \{j\}$, but it can be equivalently defined as a summation of all orders of players:

$$\phi_j = \frac{1}{|P|!} \sum_{\tau \in \Pi} \left(v(S_j^\tau \cup \{j\}) - v(S_j^\tau) \right)$$

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  $\Rightarrow$ Example: Players $1, 2, 3 \Rightarrow \Pi = \{(1,2,3), (1,3,2), (2,1,3), (2,3,1), (3,1,2), (3,2,1)\}$

  $\rightsquigarrow$ For order $\tau = (2,1,3)$ and player of interest $j = 3 \Rightarrow S_j^\tau = \{2,1\}$

  $\rightsquigarrow$ For order $\tau = (3,1,2)$ and player of interest $j = 1 \Rightarrow S_j^\tau = \{3\}$

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- Order definition: Marginal contribution of orders that yield set $S = \{1, 2\}$ is summed twice

  $\rightsquigarrow$ In set definition, it has the weight $\frac{2!(3-2-1)!}{3!} = \frac{2 \cdot 0!}{6} = \frac{2}{6}$

SHAPLEY VALUE - COMMENTS ON ORDER DEFINITION

- Order and set definition are equivalent
- Reason: The number of orders which yield the same coalition S is $|S|!(|P| - |S| - 1)!$
  - $\Rightarrow$ There are $|S|!$ possible orders of players within coalition S
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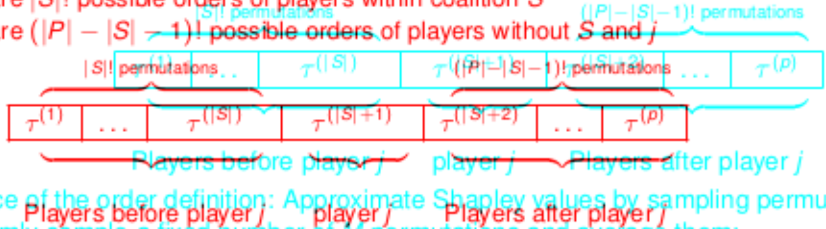

- Relevance of the order definition: Approximate Shapley values by sampling permutations
  $\rightsquigarrow$ randomly sample a fixed number of M permutations and average them:

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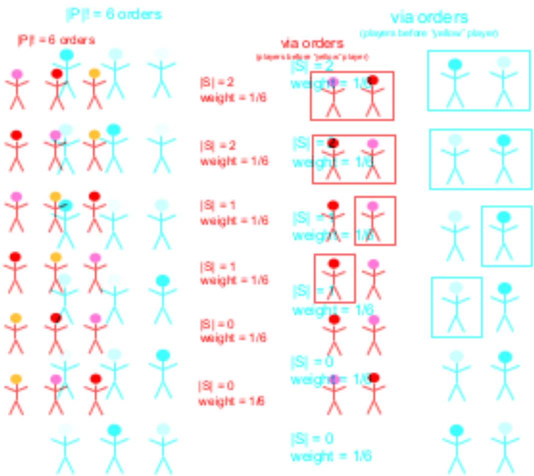

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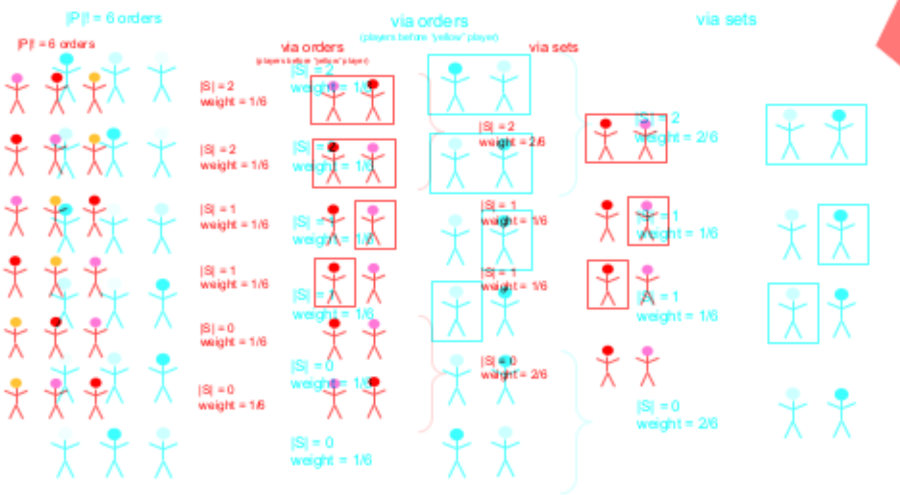

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SHAPLEY VALUES - ILLUSTRATION

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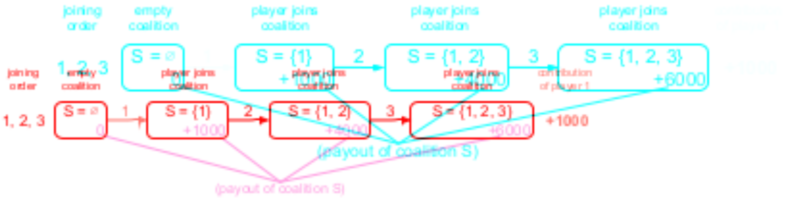
- Produce all possible joining orders of player coalitions



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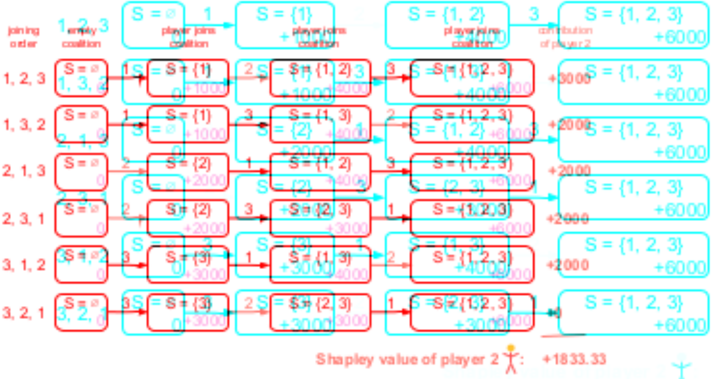

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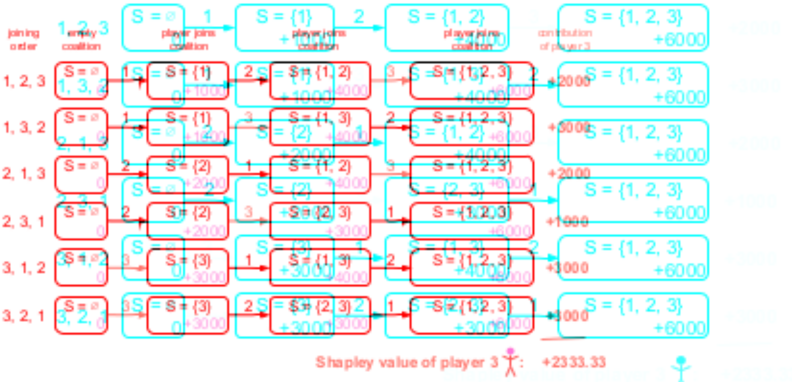

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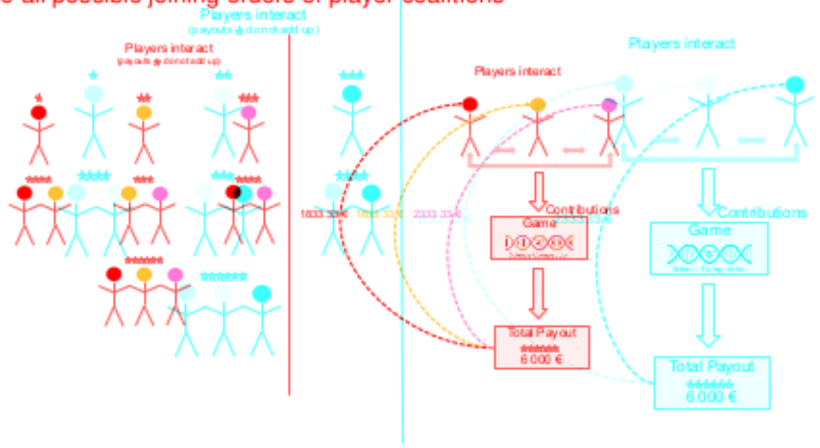

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AXIOMS OF FAIR PAYOUTS

Why is this a fair payout solution?

One possibility to define fair payouts are the following axioms for a given value function v:

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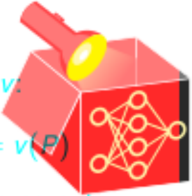

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- **Additivity**: For a game v with combined payouts $v(S) = v_1(S) + v_2(S)$, the payout is the sum of payouts: $\phi_{j,v} = \phi_{j,v_1} + \phi_{j,v_2}$

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