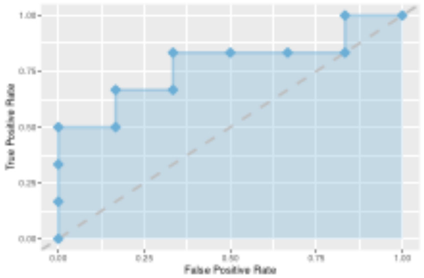
AUC AS A RANK-BASED METRIC

- The AUC metric is intimately related to the **Mann-Whitney-U test**, also known as **Wilcoxon rank-sum test**.
- This connection is best understood viewing the AUC from a slightly different angle: it is, in effect, a **rank-based** metric.
- Recall that, constructing the ROC curve, we count TP and FP.



- The AUC abstracts from the actual classification scores and considers only their rank.

MANN-WHITNEY-U TEST

- The Mann-Whitney-U test is a **non-parametric hypothesis test** on the difference in location between two samples X_1, X_2 of sizes n_1 and n_2, respectively.
- Under the null, X_1 and X_2 follow the same (unknown) distribution $\mathbb{P}$, and for any pair of observations $x_{1,1} \in X_1$, $x_{2,1} \in X_2$ drawn at random from $\mathbb{P}$, the following statement holds: $\mathbb{P}(x_{1,1} \in X_1) > \mathbb{P}(x_{2,1} \in X_2) = \mathbb{P}(x_{1,1} \in X_1) < \mathbb{P}(x_{2,1} \in X_2) = 0.5$.
- The test statistic estimates the probability of a random sample from X_1 ranking higher than one from X_2 (R_1 denoting the sum of ranks of the $x_{1,i}$):

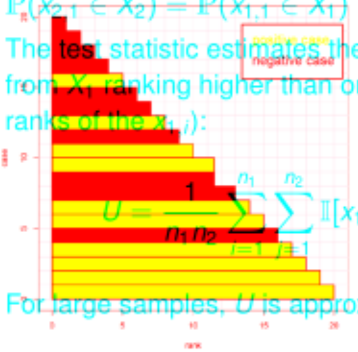
$$U = \frac{1}{n_1 n_2} \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \mathbb{I}[x_{1,i} > x_{2,j}] = R_1 - \frac{n_1(n_1+1)}{2}$$

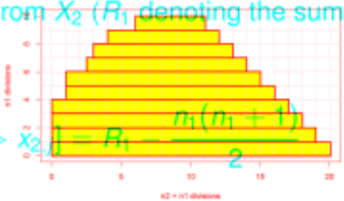
- For large samples, U is approximately normally distributed.

- We can interpret the AUC as the probability of our classifier ranking a random positive observation higher than a random negative one.
- A perfect classifier will rank all positive above all negative observations, achieving AUC = 1.

AUC & MANN-WHITNEY-U TEST

- We can directly interpret the AUC in the light of the U statistic.
- In order to see this, plot the ranks of all the scores as a stack of horizontal bars, and color them by label.
- Next, keep only the green ones, and slide them horizontally to get a nice even stairstep on the right edge:





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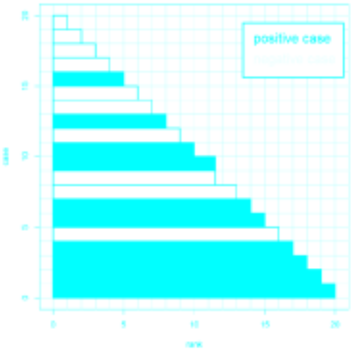
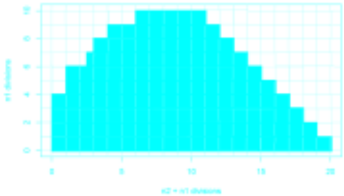
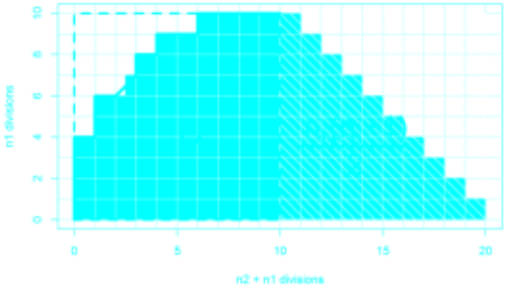


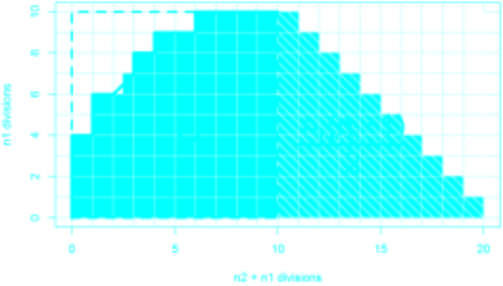


AUC & MANN-WHITNEY-U TEST



- Denoting the respective numbers of cases as n_+ and n_-, we have:
  $U = R_+ - \frac{n_+(n_+ + 1)}{2}$.
- The area of the green bars on the right is equal to $\frac{n_+(n_+ + 1)}{2}$.

AUC & MANN-WHITNEY-U TEST



- U: area of the green bars on the left.
- $n_+ \cdot n_-$: area of the dashed rectangle.

$\Rightarrow$ AUC is U normalized to the unit square:

$$\text{AUC} = \frac{U}{n_+ \cdot n_-}.$$