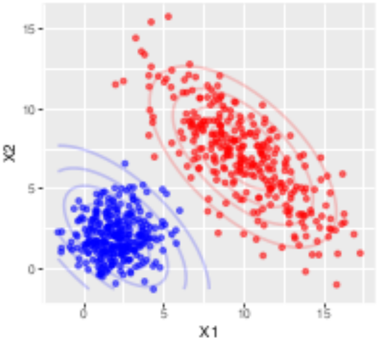
LINEAR DISCRIMINANT ANALYSIS (LDA) / 2

LDA assumes that each class density is modeled as a *multivariate Gaussian*:

$$p(\mathbf{x}|y=k) = \frac{1}{(2\pi)^{\frac{p}{2}}|\Sigma|^{\frac{1}{2}}} \exp\left(-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu}_k)^T\Sigma^{-1}(\mathbf{x}-\boldsymbol{\mu}_k)\right)$$

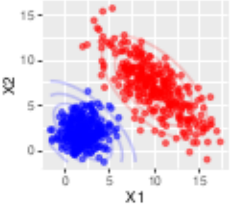
with equal covariance, i. e. $\Sigma_k = \Sigma \quad \forall k$.



LINEAR DISCRIMINANT ANALYSIS (LDA) / 3

Parameters θ are estimated in a straightforward manner by estimating

$$
\begin{aligned}
\hat{\pi}_k &= \frac{n_k}{n}, \text{ where } n_k \text{ is the number of class-}k\text{ observations} \\
\hat{\boldsymbol{\mu}}_k &= \frac{1}{n_k} \sum_{i: y^{(i)} = k} \mathbf{x}^{(i)} \\
\hat{\Sigma} &= \frac{1}{n-g} \sum_{k=1}^{g} \sum_{i: y^{(i)} = k} (\mathbf{x}^{(i)} - \hat{\boldsymbol{\mu}}_k)(\mathbf{x}^{(i)} - \hat{\boldsymbol{\mu}}_k)^T
\end{aligned}
$$



LDA AS LINEAR CLASSIFIER / 2

We can formally show that LDA is a linear classifier, by showing that the posterior probabilities can be written as linear scoring functions - up to any isotonic / rank-preserving transformation.

$$\pi_k(\mathbf{x}) = \frac{\pi_k \cdot p(\mathbf{x}|y=k)}{p(\mathbf{x})} = \frac{\pi_k \cdot p(\mathbf{x}|y=k)}{\sum\limits_{j=1}^{g} \pi_j \cdot p(\mathbf{x}|y=j)}$$

As the denominator is the same for all classes we only need to consider

$$\pi_k \cdot p(\mathbf{x}|y=k)$$

and show that this can be written as a linear function of $\mathbf{x}$.

LDA AS LINEAR CLASSIFIER / 3

$$
\begin{aligned}
& \pi_k \cdot p(\mathbf{x}|y=k) \\
\propto \quad & \pi_k \exp\left(-\frac{1}{2}\mathbf{x}^T\Sigma^{-1}\mathbf{x} - \frac{1}{2}\boldsymbol{\mu}_k^T\Sigma^{-1}\boldsymbol{\mu}_k + \mathbf{x}^T\Sigma^{-1}\boldsymbol{\mu}_k\right) \\
= \quad & \exp\left(\log \pi_k - \frac{1}{2}\boldsymbol{\mu}_k^T\Sigma^{-1}\boldsymbol{\mu}_k + \mathbf{x}^T\Sigma^{-1}\boldsymbol{\mu}_k\right)\exp\left(-\frac{1}{2}\mathbf{x}^T\Sigma^{-1}\mathbf{x}\right) \\
= \quad & \exp\left(\boldsymbol{\theta}_{0k} + \mathbf{x}^T\boldsymbol{\theta}_k\right)\exp\left(-\frac{1}{2}\mathbf{x}^T\Sigma^{-1}\mathbf{x}\right) \\
\propto \quad & \exp\left(\boldsymbol{\theta}_{0k} + \mathbf{x}^T\boldsymbol{\theta}_k\right)
\end{aligned}
$$

by defining $\boldsymbol{\theta}_{0k} := \log \pi_k - \frac{1}{2}\boldsymbol{\mu}_k^T\Sigma^{-1}\boldsymbol{\mu}_k$ and $\boldsymbol{\theta}_k := \Sigma^{-1}\boldsymbol{\mu}_k$.

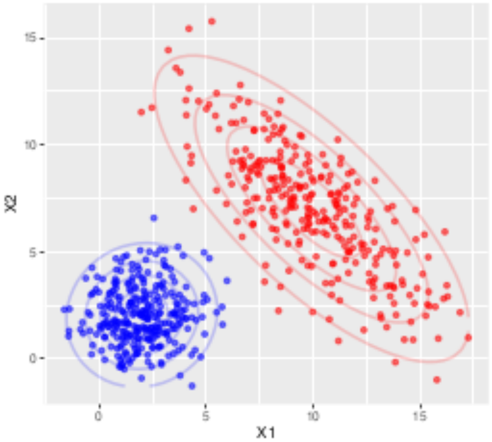
We have again left out all constants which are the same for all classes k, so the normalizing constant of our Gaussians and $\exp\left(-\frac{1}{2}\mathbf{x}^T\Sigma^{-1}\mathbf{x}\right)$.

By finally taking the log, we can write our transformed scores as linear:

$$f_k(\mathbf{x}) = \boldsymbol{\theta}_{0k} + \mathbf{x}^T\boldsymbol{\theta}_k$$

QUADRATIC DISCRIMINANT ANALYSIS (QDA) / 2

- Covariance matrices can differ over classes.
- Yields better data fit but also requires estimation of more parameters.



QUADRATIC DISCRIMINANT ANALYSIS (QDA) / 3

$$
\begin{aligned}
\pi_k(\mathbf{x}) &\propto \pi_k \cdot p(\mathbf{x}|y=k) \\
&\propto \pi_k |\Sigma_k|^{-\frac{1}{2}} \exp(-\frac{1}{2}\mathbf{x}^T\Sigma_k^{-1}\mathbf{x} - \frac{1}{2}\boldsymbol{\mu}_k^T\Sigma_k^{-1}\boldsymbol{\mu}_k + \mathbf{x}^T\Sigma_k^{-1}\boldsymbol{\mu}_k)
\end{aligned}
$$

Taking the log of the above, we can define a discriminant function that is quadratic in x.

$$
\log \pi_k - \frac{1}{2}\log|\Sigma_k| - \frac{1}{2}\boldsymbol{\mu}_k^T\Sigma_k^{-1}\boldsymbol{\mu}_k + \mathbf{x}^T\Sigma_k^{-1}\boldsymbol{\mu}_k - \frac{1}{2}\mathbf{x}^T\Sigma_k^{-1}\mathbf{x}
$$

QUADRATIC DISCRIMINANT ANALYSIS (QDA) / 4