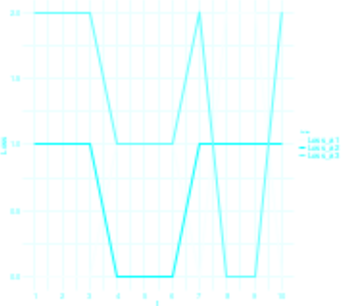
THE ONLINE LEARNER

- In the following, we will consider a first (online) learner for online learning problems. Note that a learner can be defined in a formal way.

Advanced Machine Learning

Simple Online Learning Algorithms



Learning goals

- Formalization of online learning algorithms
- Getting to know the FTL algorithm
- See that it works for online quadratic optimzation (OQO) problems

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- Indeed, a learner within the basic online learning protocol, say `Algo`, is a function

$$A : \bigcup_{t=1}^{T} (\mathcal{Z} \times \mathcal{A})^t \to \mathcal{A}$$

that returns the current action based on (the loss and) the full history of information so far:

$$a_{t+1}^{\texttt{Algo}} = A(z_1, a_1^{\texttt{Algo}}, z_2, a_2^{\texttt{Algo}}, \ldots, z_t, a_t^{\texttt{Algo}};).$$

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- In the extended online learning scenario, where the environmental data consists of two parts, $z_t = (z_t^{(1)}, z_t^{(2)})$, and the first part is revealed before the action in t is performed, we have that

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- It will be desired that the online learner admits a *cheap update formula*, which is incremental, i.e., only a portion of the previous data is necessary to determine the next action.
- Indeed, a learner within the basic online learning protocol, say Algo, is a function

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- For instance, there exists a function $u : \mathcal{Z} \times \mathcal{A} \to \mathcal{A}$ such that

$$A(z_1, a_1^{\mathtt{Algo}}, z_2, a_2^{\mathtt{Algo}}, \ldots, z_t, a_t^{\mathtt{Algo}};) = u(z_t, a_t^{\mathtt{Algo}}).$$

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FOLLOW THE LEADER ALGORITHM

- A simple algorithm to tackle online learning problems is the **Follow the leader** (FTL) algorithm.
- The algorithm takes as its action $a_t^{FTL} \in \mathcal{A}$ in time step $t \geq 2$, the element which has the minimal cumulative loss so far over the previous $t-1$ time periods:

$$a_t^{FTL} \in \arg\min_{a \in \mathcal{A}} \sum_{s=1}^{t-1} \ell(a, z_s).$$

(Technical side note: if there are more than one minimum, then one of them is chosen. Moreover, a_1^{FTL} is arbitrary.)

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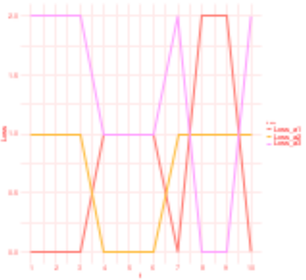
- *Interpretation:* The action a_t^{FTL} is the current "leader" of the actions in $\mathcal{A}$ in time step t, as it has the smallest cumulative loss (error) so far.



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- Note that the action selection rule of FTL is natural and has much in common with the classical batch learning approaches based on empirical risk minimization.
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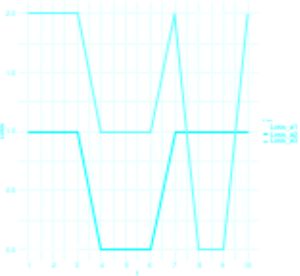

FTL: A HELPFUL LEMMA

Lemma: Let $a_1^{\mathrm{FTL}}, a_2^{\mathrm{FTL}}, \ldots$ be the sequence of actions used by the FTL algorithm for the environmental data sequence $z_1, z_2, \ldots$.

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\begin{aligned}
R_T^{\mathrm{FTL}}(\tilde{a}) &= \sum_{t=1}^{T} \left((a_t^{\mathrm{FTL}}, z_t) - (\tilde{a}, z_t) \right) \\
&\leq \sum_{t=1}^{T} \left((a_t^{\mathrm{FTL}}, z_t) - (a_{t+1}^{\mathrm{FTL}}, z_t) \right) \\
&= \sum_{t=1}^{T} (a_t^{\mathrm{FTL}}, z_t) - \sum_{t=1}^{T} (a_{t+1}^{\mathrm{FTL}}, z_t).
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In particular,

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Interpretation: the regret of the FTL algorithm is bounded by the difference of cumulated losses of itself compared to its one-step lookahead cheater version.

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Proof: In the following, we denote $a_1^{\mathrm{FTL}}, a_2^{\mathrm{FTL}}, \ldots$ simply by $a_1, a_2, \ldots$.

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$$R_T^{\mathrm{FTL}}(\tilde{a}) = \sum_{t=1}^{T} \big((a_t, z_t) - (\tilde{a}, z_t)\big) \le \sum_{t=1}^{T} \big((a_t, z_t) - (a_{t+1}, z_t)\big)$$

$$\Leftrightarrow \sum_{t=1}^{T} (a_{t+1}, z_t) \le \sum_{t=1}^{T} (\tilde{a}, z_t).$$

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$$\Leftrightarrow \sum_{t=1}^{T} L(a_{t+1}, z_t) \leq \sum_{t=1}^{T} L(\tilde{a}, z_t).$$

Hence, we will verify the inequality $\sum_{t=1}^{T} (a_{t+1}, z_t) \leq \sum_{t=1}^{T} (\tilde{a}, z_t)$, which implies the assertion.

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Reminder: $a_t^{\mathrm{FTL}} \in \arg\min_{a \in \mathcal{A}} \sum_{s=1}^{t-1} (a, z_s).$

Initial step: $T = 1$. It holds that

$$\sum_{t=1}^{T} (a_{t+1}, z_t) = (a_2, z_1) = \left(\arg\min_{a \in \mathcal{A}} (a, z_1), z_1 \right)$$

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Then, the following holds as well (adding (a_{T+1}, z_T) on both sides)

$$\sum_{t=1}^{T} (a_{t+1}, z_t) \le (a_{T+1}, z_T) + \sum_{t=1}^{T-1} (\tilde{a}, z_t), \quad \forall \tilde{a} \in \mathcal{A}.$$

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for all $\tilde{a} \in \mathcal{A}$. □

FTL FOR OQO PROBLEMS

- One popular instantiation of the online learning problem is the problem of *online quadratic optimization* (OQO).
- In its most general form, the loss function is thereby defined as

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- **Proposition:** Using FTL on any online quadratic optimization problem with $\mathcal{A} = \mathbb{R}^d$ and $V = \sup_{z \in \mathcal{Z}} ||z||_2$, leads to a regret of

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- This result is satisfactory for three reasons:
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