

FTL FOR OQO PROBLEMS: ANALYSIS

- One popular instantiation of the online learning problem is the problem of *online quadratic optimization* (OQO).
- **Proof:**
- In its most general form, the loss function is thereby defined as
 - In the following, we denote $a_1^{\text{FTL}}, a_2^{\text{FTL}}, \dots$ simply by a_1, a_2, \dots

$$L(a_t, z_t) = \frac{1}{2} \|a_t - z_t\|_2^2.$$

where $\mathcal{A}, \mathcal{Z} \subset \mathbb{R}^d$.

- **Proposition:** Using FTL on any online quadratic optimization problem with $\mathcal{A} = \mathbb{R}^d$ and $V = \sup_{z \in \mathcal{Z}} \|z\|_2$, leads to a regret of

$$R_T^{\text{FTL}} \leq 4V^2 (\log(T) + 1).$$



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- In the following, we denote $a_1^{\text{FTL}}, a_2^{\text{FTL}}, \dots$ simply by a_1, a_2, \dots

Reminder (Useful Lemma):

$$R_T^{\text{FTL}} \leq \sum_{t=1}^T (a_t^{\text{FTL}}, z_t) - \sum_{t=1}^T (a_{t+1}^{\text{FTL}}, z_t)$$



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Reminder (Useful Lemma):

$$F_T^{\text{FTL}} \leq \sum_{t=1}^T L(a_t^{\text{FTL}}, z_t) - \sum_{t=1}^T L(a_{t+1}^{\text{FTL}}, z_t)$$

- Using this lemma, we just have to show that

$$\sum_{t=1}^T ((a_t, z_t) - (a_{t+1}, z_t)) \leq 4L^2 \cdot (\log(T) + 1). \quad (1)$$



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- So, we will prove (1). For this purpose, we compute the explicit form of the actions of FTL for this type of online learning problem.



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- **Claim:** It holds that $a_t = \frac{1}{t-1} \cdot \sum_{s=1}^{t-1} z_s$, if $(a, z) = \frac{1}{2} \|a - z\|_2^2$.
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- Claim: It holds that $a_t = \frac{1}{t-1} \cdot \sum_{s=1}^{t-1} z_s$, if $(a, z) = \frac{1}{2} \|a - z\|_2^2$.
 - Recall that

$$a_t^{\text{FTL}} = \arg \min_{a \in \mathcal{A}} \sum_{s=1}^{t-1} (a, z_s) = \arg \min_{a \in \mathcal{A}} \sum_{s=1}^{t-1} \frac{1}{2} \|a - z_s\|_2^2.$$



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- So, we have to find the minimizer of the function

$$f(a) := \sum_{s=1}^{t-1} \frac{1}{2} \|a - z_s\|_2^2 = \sum_{s=1}^{t-1} \frac{1}{2} (a - z_s)^\top (a - z_s).$$



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- Compute $\nabla f(a) = \sum_{s=1}^{t-1} a - z_s = (t-1)a - \sum_{s=1}^{t-1} z_s$, which we set to zero and solve with respect to a to obtain the claim.

(f is convex, so that this leads indeed to a minimizer.)



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- Hence, a_t is the empirical average of $z_1(a, z), z_2(a, z), \dots, z_t(a, z)$ and we can provide the following incremental update formula for its computation

$$a_t^{\text{FTL}} = \arg \min_{a \in \mathcal{A}} \sum_{s=1}^{t-1} L(a, z_s) = \arg \min_{a \in \mathcal{A}} \sum_{s=1}^{t-1} \frac{1}{2} \|a - z_s\|_2^2.$$
$$a_{t+1} = \frac{1}{t} \cdot \sum_{s=1}^t z_s = \frac{1}{t} \left(z_t + \sum_{s=1}^{t-1} z_s \right)$$

- So, we have to find the minimizer of the function
- $$= \frac{1}{t} (z_t + (t-1)a_t) = \frac{1}{t} z_t + \left(1 - \frac{1}{t}\right) a_t.$$

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$$\begin{aligned} a_{t+1} &= \frac{1}{t} \cdot \sum_{s=1}^t z_s = \frac{1}{t} \left(z_t + \sum_{s=1}^{t-1} z_s \right) \\ &= \frac{1}{t} (z_t + (t-1)a_t) = \frac{1}{t} z_t + \left(1 - \frac{1}{t}\right) a_t. \end{aligned}$$

- From the last display we derive that

$$a_{t+1} - z_t = \left(1 - \frac{1}{t}\right) \cdot a_t + \frac{1}{t} z_t - z_t = \left(1 - \frac{1}{t}\right) \cdot (a_t - z_t).$$



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- Claim:

$$(a_t, z_t) - (a_{t+1}, z_t) \leq \frac{1}{t} \cdot \|a_t - z_t\|_2^2. \quad (2)$$



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- Hence, a_t is the empirical average of z_1, \dots, z_{t-1} and we can provide the following incremental update formula for its computation
- Indeed, this can be seen as follows

$$\begin{aligned} a_{t+1} &= \frac{1}{t} \cdot \left(\sum_{s=1}^t z_s \right) \\ &= \frac{1}{t} \left(\sum_{s=1}^{t-1} z_s + z_t \right) \\ &= \frac{1}{t} \left((t-1) a_t + z_t \right) \\ &= \left(1 - \frac{1}{t} \right) a_t + \frac{1}{t} z_t \end{aligned}$$

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$$\begin{aligned}L(a_t, z_t) - L(a_{t+1}, z_t) &= \frac{1}{2} \|a_t - z_t\|_2^2 - \frac{1}{2} \|a_{t+1} - z_t\|_2^2 \\ &= \frac{1}{2} \left(\|a_t - z_t\|_2^2 - \|a_{t+1} - z_t\|_2^2 \right) \\ &= \frac{1}{2} \left(\|a_t - z_t\|_2^2 - \left\| \left(1 - \frac{1}{t}\right) \cdot (a_t - z_t) \right\|_2^2 \right).\end{aligned}$$

- And from this,

$$\begin{aligned}(a_t, z_t) - (a_{t+1}, z_t) &= \frac{1}{2} \left(\|a_t - z_t\|_2^2 - \left(1 - \frac{1}{t}\right)^2 \cdot \|a_t - z_t\|_2^2 \right) \\ &= \frac{1}{2} \left(1 - \left(1 - \frac{1}{t}\right)^2 \right) \cdot \|a_t - z_t\|_2^2 \\ &= \left(\frac{1}{t} - \frac{1}{2t^2} \right) \cdot \|a_t - z_t\|_2^2 \\ &\leq \frac{1}{t} \cdot \|a_t - z_t\|_2^2.\end{aligned}$$



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- Since by assumption $L = \sup_{z \in \mathcal{Z}} \|z\|_2$ and a_t is the empirical average of z_1, \dots, z_{t-1} , we have that $\|a_t\|_2 \leq L$.



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- Now the triangle inequality states that for any two vectors $x, y \in \mathbb{R}^d$ it holds that

$$\|x + y\|_2 \leq \|x\|_2 + \|y\|_2,$$

so that

$$\|a_t - z_t\|_2 \leq \|a_t\|_2 + \|z_t\|_2 \leq 2L. \quad (3)$$



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- Summing over all t in (2) and using (3) we arrive at

$$\begin{aligned} \sum_{t=1}^T ((a_t, z_t) - (a_{t+1}, z_t)) &\leq \sum_{t=1}^T \left(\frac{1}{t} \cdot \|a_t - z_t\|_2^2 \right) \leq \sum_{t=1}^T \frac{1}{t} \cdot (2L)^2 \\ &= 4L^2 \cdot \sum_{t=1}^T \frac{1}{t}. \end{aligned}$$



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- Now, it holds that $\sum_{t=1}^T \frac{1}{t} \leq \log(T) + 1$, so that we obtain

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which is what we wanted to prove. \square



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