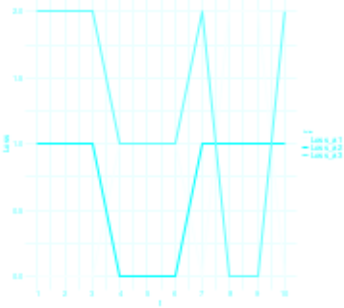
FTL FOR ONLINE LINEAR OPTIMIZATION

- Another popular instantiation of the online learning problem is the online linear optimization problem, which is characterized by a linear loss function $(a, z) = a^\top z$.

Advanced Machine Learning

Follow the leader on OLO problems



Learning goals

- Getting to know online linear optimzation (OLO) problems
- See that FTL might fail for these problems
- Understanding the root cause for FTL's flaw

FTL FOR ONLINE LINEAR OPTIMIZATION

- Another popular instantiation of the online learning problem is the online linear optimization problem, which is characterized by a linear loss function $\ell(a, z) = a^\top z$.

- Let $\mathcal{A} = [-1, 1]$ and suppose that $z_t = \begin{cases} -\frac{1}{2}, & t = 1, \\ 1, & t \text{ is even}, \\ -1, & t \text{ is odd}. \end{cases}$

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- No matter how we choose the first action a_1^{FTL}, it will hold that FTL has a cumulative loss greater than (or equal) $T - 3/2$, while the best action in hindsight has a cumulative loss of $-1/2$.

- Thus, FTL's cumulative regret is at least $T - 1$, which is linearly growing in T.

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- Another popular instantiation of the online learning problem is the online linear optimization problem, which is characterized by a linear loss function $L(a, z) = a\,z$.
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- No matter how we choose the first action a_1^{FTL}, it will hold that FTL has a cumulative loss greater than (or equal) $T - 3/2$, while the best action in hindsight has a cumulative loss of $-1/2$.
- Thus, FTL's cumulative regret is at least $T - 1$, which is linearly growing in T.

- Indeed, note that

$$a_{t+1}^{\mathrm{FTL}} = \arg\min_{a \in \mathcal{A}} \sum_{s=1}^{t} (a, z_s) = \arg\min_{a \in [-1,1]} a \sum_{s=1}^{t} z_s = \begin{cases} -1, & \text{if } \sum_{s=1}^{t} z_s > 0, \\ 1, & \text{if } \sum_{s=1}^{t} z_s < 0, \\ \text{arbitrary}, & \text{if } \sum_{s=1}^{t} z_s = 0. \end{cases}$$

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t	a_t^{FTL}	z_t	$(a_t^{\mathrm{FTL}}, z_t)$	$\sum_{s=1}^{t} (a_s^{\mathrm{FTL}}, z_s)$	$\sum_{s=1}^{t} z_s$
1	1	$-1/2$	$-1/2$	$-1/2$	$-1/2$

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1	1	$-1/2$	$-1/2$	$-1/2$	$-1/2$
2	1	1	1	$1 - 1/2$	$1/2$

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t	a_t^{FTL}	z_t	$\langle a_t^{\mathrm{FTL}}, z_t \rangle$	$\sum_{s=1}^{t} \langle a_s^{\mathrm{FTL}}, z_s \rangle$	$\sum_{s=1}^{t} z_s$
1	1	$-1/2$	$-1/2$	$-1/2$	$-1/2$
2	1	1	1	$1 - 1/2$	$1/2$
3	-1	-1	1	$2 - 1/2$	$-1/2$

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3	-1	-1	1	$2-1/2$	$-1/2$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
T	$(-1)^T$	$(-1)^T$	1	$T-1-1/2$	$(-1/2)^T$

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$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
T	$(-1)^T$	$(-1)^T$	1	$T-1-1/2$	$(-1/2)^T$

- The best action has cumulative loss

$$\inf_{a\in\mathcal{A}} \sum_{s=1}^{T} \langle a, z_s\rangle = \inf_{a\in[-1,1]} a \underbrace{\sum_{s=1}^{T} z_s}_{=(-1/2)^T} = -1/2.$$

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- Thus, we see: FTL can fail for online linear optimization problems, although it is well suited for online quadratic optimization problems!
- The reason is that the action selection of FTL is not stable enough (caused by the loss function), which is fine for the latter problem, but problematic for the former.
- One has to note that the online linear optimization problem example above, where FTL fails, is in fact an adversarial learning setting: The environmental data is generated in such a way that the FTL learner is fooled in each time step.
- The best action has cumulative loss

$$\inf_{a \in \mathcal{A}} \sum_{s=1}^{T} L(a, z_s) = \inf_{a \in [-1,1]} a \underbrace{\sum_{s=1}^{T} z_s}_{=(-1/2)^T} = -1/2.$$

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